Star Formation 2020

Q&A 05.05.2020

Worked Example - Isothermal sphere

Interstellar gas clouds are subject to their own gravitational force leading to a constant compression of the cloud. This is countered gas pressure, which is increasing with increasing gas density. Consider the density structure and the gravitational potential of such a cloud in equilibrium.

We assume a spherical, non-turbulent cloud with a constant gas temperature. Three equations govern the equilibrium configuration:

The equation of the hydrostatic equilibrium:

$$-\nabla \Phi_g - \frac{\nabla P}{\rho} = 0 \tag{1}$$

(ρ: density, P pressure, Φ_g: gravitational potential)

The state equation of an isothermal gas:

$$P = \rho \, a_t^2 \tag{2}$$

(a_t: sound speed)

With, Φ_g obeying the Poisson equation:

$$\nabla^2 \Phi_g = 4\pi G \rho \tag{3}$$

(G: gravitational constant)

a) Lane-Emden equation

Substituting (2) into (1) and applying physically reasonable boundary conditions, we find for the spherical case:

$$\rho(r) = \rho_c \exp\left(-\frac{\Phi_g}{a^2}\right) \tag{4}$$

(ρ_c: central density (r=0))

With equation (4) follows for the Poisson-equation (3) in the spherical case:

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\Phi_g}{dr}\right) == 4\pi G \rho_c \, e^{-\frac{\Phi_g}{a^2}} \tag{5}$$

Find the correct form to substitute Φ_g with the dimensionless potential ϕ and the radius r with the dimensionless length ξ, such that the following dimensionless form of (5) holds:

Substituting $\phi = \frac{\Phi_g}{a^2}$ and $\xi = \left(\frac{4\pi G \rho_c}{a^2}\right)^{1/2} r$ we find

$$\frac{1}{\xi^2} \frac{1}{d\xi}\left(\xi^2 \frac{d\phi}{d\xi}\right) = e^{-\phi} \tag{6}$$

Equation (69 is called *Lane-Emden equation*.

Dimensionless formalism

The dimensionless form of the Bonnor Ebert Sphere is

In[ ]:= **LaneEmdenGleichung = $\frac{1}{\xi^2}$ D$\left[\xi^2 \varphi'[\xi], \xi\right]$ == Exp[-$\varphi[\xi]$] // Simplify**

Out[ ]= $\frac{2\,\varphi'[\xi]}{\xi} + \varphi''[\xi] == e^{-\varphi[\xi]}$

In[ ]:= **solution = With$\left[\left\{c = 10^{-50}\right\},\right.$**
    NDSolve[{LaneEmdenGleichung, $\varphi[c]$ == 0, $\varphi'[c]$ == 0}, φ, {ξ, c, 100}][[1, 1, 2]]$\right]$

Out[ ]= InterpolatingFunction$\left[\begin{array}{c}\boxed{+}\end{array}\begin{array}{c}\text{Domain: } \{\{1.\times 10^{-50}, 100.\}\}\\\text{Output: scalar}\end{array}\right]$

In[ ]:= **Plot$\left[\{Exp[-solution[x]], solution[x]\}, \{x, .1, 7\}, Frame \to True,\right.$**
    PlotLegends $\to \left\{\frac{\rho[\xi]}{\rho_0}, \varphi_g[\xi]\right\}$, FrameLabel $\to \{\xi, None\}$,
    Epilog $\to \left\{Text\left[\frac{\rho[\xi]}{\rho_0}, \{1, 1.2\}\right], Text[\varphi_g[\xi], \{4, 1.2\}]\right\}\right]$

Out[ ]=
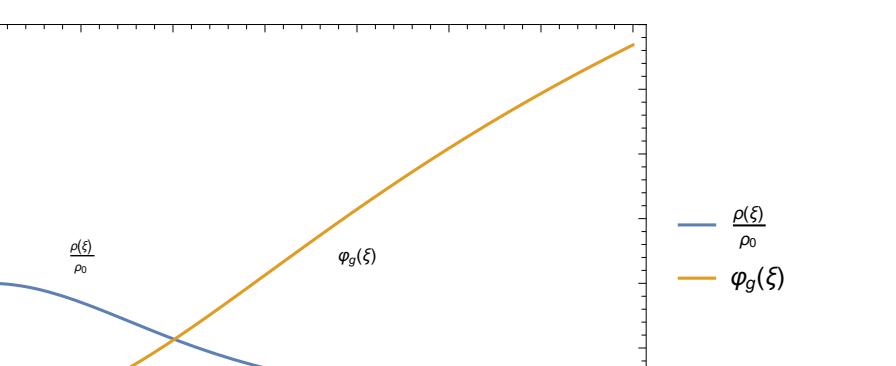
with dimensions

In[ ]:= **LaneEmdenGleichung2 = $\frac{1}{r^2}$ D$\left[r^2 \phi'[r], r\right]$ == 4 π G ρ0 Exp$\left[-\frac{\phi[r]}{a^2}\right]$ // Simplify**

Out[ ]= $\frac{2\,\phi'[r]}{r} + \phi''[r] == 4 e^{-\frac{\phi[r]}{a^2}} G \pi \rho 0$

In[ ]:= **constants = $\left\{G \to 6.6726 \times 10^{-8}, a \to 5 \times 10^5, \rho 0 \to 1.2 \times 10^{17}\right\}$**

Out[ ]= $\left\{G \to 6.6726 \times 10^{-8}, a \to 500\,000, \rho 0 \to 1.2 \times 100\,000\,000\,000\,000\,000\right\}$

In[ ]:= `solution2 = With[{c = 10^-10,`
    `NDSolve[{LaneEmdenGleichung2 /. {a → 10^5, G → 6.6726 × 10^-8, ρ0 → 10^6},`
      `φ[c] == 0, φ'[c] == 0}, φ, {r, c, 10^10}][[1, 1, 2]]]`

Out[ ]= InterpolatingFunction[⊞ ⟋ Domain: {{1. × 10⁻¹⁰, 1. × 10¹⁰}} Output: scalar]

In[ ]:= `LogLogPlot[{10^6 Exp[- solution2[x]/10^10], solution2[x]}, {x, 10, 10^8},`
    `Frame → True, FrameLabel → {r / cm, None}, PlotLegends → {ρ[r], φg[r]},`
    `Epilog → {Text[ρ[r], Log@{100, 10^7}], Text[φg[r], {4, 1.2}]}]`

Out[ ]=
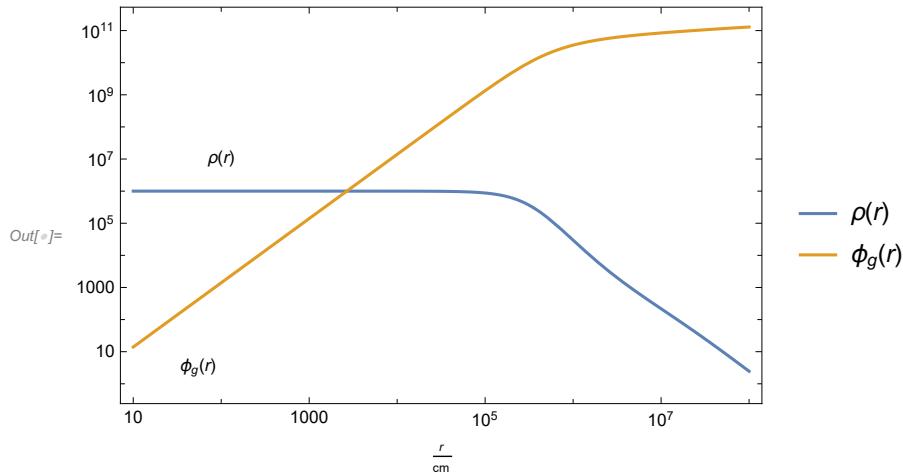
b) Discretization

Equation (6) can not be solved analytically. To get the density structure and the gravitational potential one has to integrate (6) numerically. There are a number of different methods to do this. We consider here the discretization via finite differences where differential operators are replaced by discrete, approximate expressions which are calculated at discrete grid positions, e.g.:

$$y'(x) = \frac{dy(x)}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y_n - y_{n-1}}{\Delta x}, \quad n = 1, \ldots, N \tag{7}$$

Find a suitable discretization for the differential equation (6) and show that the following is true:

$$\phi_n = \Delta r^2 \, e^{-(\phi_{n-1})} + \frac{2(n-2)\,\phi_{n-1} - (n-3)\,\phi_{n-2}}{n-1} \tag{8}$$

Δr is the (constant) distance between the discrete radius grid points.

Numerical Solution

In[ ]:= `Clear[φ]`

In[ ]:= `1/ξ^2 D[ξ^2 φ'[ξ], ξ] == Exp[-φ[ξ]]`

Out[ ]= $\dfrac{2\,\xi\,\varphi'[\xi] + \xi^2\,\varphi''[\xi]}{\xi^2}$ == $e^{-\varphi[\xi]}$

```
discretization = {
    φ'[ξ] :> (φ[[n - 1]] - φ[[n - 2]])/Δr,
    φ[ξ] :> φ[[n - 1]],
    ξ :> (n - 1) Δr,
    φ''[ξ] :> (φ[[n]] - 2 φ[[n - 1]] + φ[[n - 2]])/(Δr)^2 }
```

Out[•]= $\left\{\varphi'[\xi] :\to \dfrac{\varphi[\![n-1]\!] - \varphi[\![n-2]\!]}{\triangle r},\ \varphi[\xi] :\to \varphi[\![n-1]\!],\right.$

$\left.\xi :\to (n-1)\,\triangle r,\ \varphi''[\xi] :\to \dfrac{\varphi[\![n]\!] - 2\,\varphi[\![n-1]\!] + \varphi[\![n]\!]}{\triangle r^2}\right\}$

In[•]:= **LaneEmdenGleichung /. discretization**

Out[•]= $\dfrac{2\left(-\varphi[\![-2+n]\!] + \varphi[\![-1+n]\!]\right)}{(-1+n)\,\triangle r^2} + \dfrac{\varphi[\![-2+n]\!] - 2\,\varphi[\![-1+n]\!] + \varphi[\![n]\!]}{\triangle r^2} == e^{-\varphi[\![-1+n]\!]}$

In[•]:= **Off[Part::pkspec1]**
Solve[LaneEmdenGleichung /. discretization, φ[[n]]]

Out[•]= $\left\{\left\{\varphi[\![n]\!] \to \dfrac{1}{-1+n} e^{-\varphi[\![-1+n]\!]} \left(-\triangle r^2 + n\,\triangle r^2 + 3\,e^{\varphi[\![-1+n]\!]}\,\varphi[\![-2+n]\!] - \right.\right.\right.$

$\left.\left.\left. e^{\varphi[\![-1+n]\!]}\,n\,\varphi[\![-2+n]\!] - 4\,e^{\varphi[\![-1+n]\!]}\,\varphi[\![-1+n]\!] + 2\,e^{\varphi[\![-1+n]\!]}\,n\,\varphi[\![-1+n]\!]\right)\right\}\right\}$

In[•]:= **Solve[LaneEmdenGleichung /. discretization, φ[[n]]] // FullSimplify**

Out[•]= $\left\{\left\{\varphi[\![n]\!] \to e^{-\varphi[\![-1+n]\!]}\,\triangle r^2 + \dfrac{-(-3+n)\,\varphi[\![-2+n]\!] + 2\,(-2+n)\,\varphi[\![-1+n]\!]}{-1+n}\right\}\right\}$

In[•]:= **Clear[φ]**
```
With[{
    Δr = 0.01},
  Nmax = (1/Δr) 10;
  φ = ConstantArray[0, IntegerPart[Nmax]];
  Do[φ[[n]] = (1/(-1 + n)) e^{-φ[[-1+n]]} (-Δr^2 + n Δr^2 + 3 e^{φ[[-1+n]]} φ[[-2+n]] -
     e^{φ[[-1+n]]} n φ[[-2+n]] - 4 e^{φ[[-1+n]]} φ[[-1+n]] + 2 e^{φ[[-1+n]]} n φ[[-1+n]]), {n, 3, Nmax}]];
  φ1 =
    φ;
```

In[•]:=
```
Clear[φ]
With[{
    Δr = 0.01},
  Nmax = (1/Δr) 10;
  φ = ConstantArray[0, IntegerPart[Nmax]];
  Do[φ[[n]] = e^{-φ[[-1+n]]} Δr^2 + (-(-3 + n) φ[[-2+n]] + 2 (-2 + n) φ[[-1+n]])/(-1 + n), {n, 3, Nmax}]];
  φ2 = φ;
```

In[ ]:= `ListPlot[{Table[{0.01 n, φ1[[n]]}, {n, 1, 1000, 10}]}]`

Out[ ]=
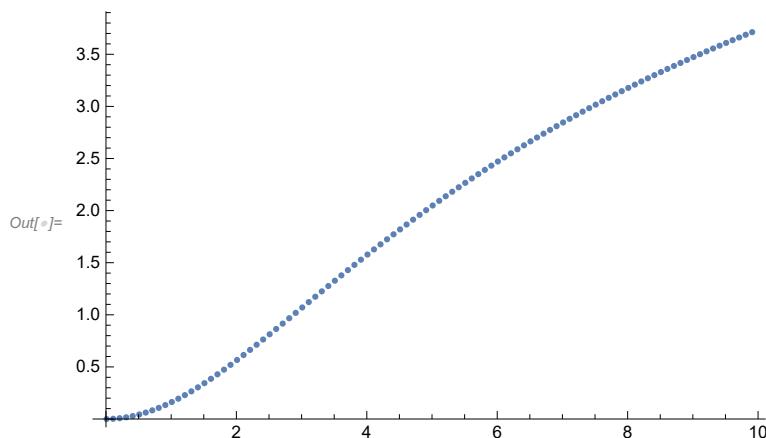
In[ ]:= `Plot[{Exp[-solution[x]], solution[x]}, {x, 0, 6},`

`Frame → True, FrameLabel → {ξ, None}, Epilog → {Text[`$\frac{\rho[\xi]}{\rho_0}$`, {1, 1.2}],`

`Text["φ_g(ξ)", {4, 1.2}], Point[Table[{0.01 n, φ1[[n]]}, {n, 1, 1000, 10}]],`

`Point[Table[{0.01 n, Exp[-φ1[[n]]]}, {n, 1, 1000, 10}]]}]`
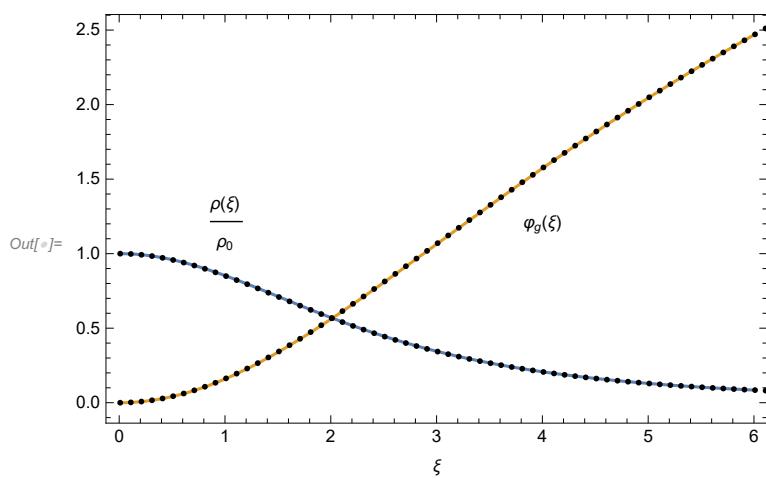
Out[ ]=

## Errors due to Δr

In[ ]:=

```
In[ ]:= Table[With[{
        Δr = j},
       Nmax = 1/Δr 10;
       φ = ConstantArray[0, IntegerPart[Nmax]];
       Do[φ[[n]] = e^(-φ[[-1+n]]) Δr² + (-(-3+n) φ[[-2+n]] + 2 (-2+n) φ[[-1+n]])/(-1+n), {n, 3, Nmax}]];
      Plot[{Exp[-solution[x]], solution[x]}, {x, 0, 6},
       Frame → True, FrameLabel → {ξ, None}, PlotLabel → "Δr=" <> ToString[j],
       Epilog → {Text[ρ[ξ]/ρ₀, {1, 1.5}], Text["φ_g(ξ)", {5, 1.4}],
         Red, Point[Table[{j n, φ[[n]]}, {n, 1, Nmax, 1}]], Blue,
         Point[Table[{j n, Exp[-φ[[n]]]}, {n, 1, Nmax, 1}]]}],
     {j, {2, 1.5, 1, 0.75, 0.5, 0.2, 0.1, 0.05}}]
    Clear[φ, Nmax]
```

Out[ ]= {

}
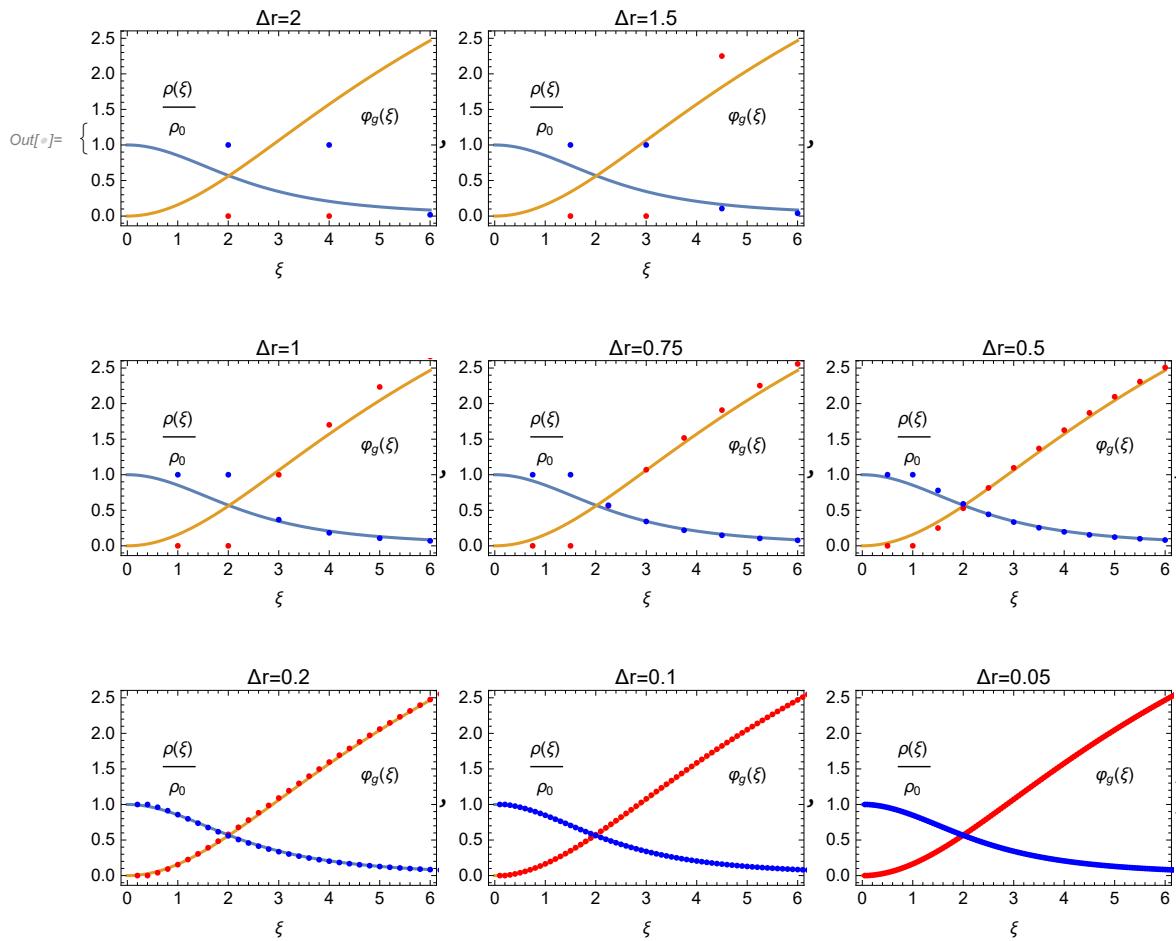
Cloud Energetics

Consider a GMC, a giant molecular cloud with

mass *M*

spherical shape with radius *R* and

constant temperature *T*. The

one-dimensional, non-thermal velocity dispersion is σ_{1D}

uniform magnetic field

the gas is a mx of hydrogen and helium in the standard cosmic abundances -> mean mass per particle $\mu = 3.9 \times 10^{-24}\,g$

Use the following typical parameters: $M = 10^6\,M_\odot$, $R = 50\,pc$, $T = 10\,K$, $\sigma_{1D} = 5\,km\,s^{-1}$, $B = 20\,\mu G$.

Calculate the gravitational energy, thermal energy, bulk kinetic energy, and magnetic energy in terms of M, R, T, σ_{1D}, and B. Evaluate the four energies and place them in in order from largest (in absolute value) to smallest for the typical GMC properties given above.

$E_{grav} = \frac{-3}{5}\frac{G M^2}{R} = 1.06 \times 10^{51}\,erg$

In[•]:= **Gconst = 6.674 $\times$ 10^{-8}; (* cm^{3}s^{-2}g^{-1} *)**
Msol = 2 $\times$ 10^{33};
pc = 3.02 $\times$ 10^{18};

$$\frac{-3}{5}\frac{\textbf{Gconst} * (10^6\,\textbf{Msol})^2}{50\,\textbf{pc}}$$

Out[•]= -1.06077×10^{51}

$E_{th} = \frac{3\,k_b\,T\,M}{2\,\mu} = 1.06 \times 10^{48}\,erg$

In[•]:= $\frac{3}{2} \times \textbf{1.38} \times 10^{-16} * \frac{10 * 10^6\,\textbf{Msol}}{3.9 \times 10^{-24}}$

Out[•]= 1.06154×10^{48}

$E_{kin} = \frac{3}{2}\,M\,\sigma_{1D}^2 = 7.46 \times 10^{50}\,erg$

In[•]:= $\frac{3}{2} \times 10^6\,\textbf{Msol}\,(5 \times 10^{5\cdot})^2$

Out[•]= 7.5×10^{50}

$E_{mag} = \frac{B^2\,R^3}{6} = 2.46 \times 10^{50}\,erg$

In[•]:= $\frac{(20 \times 10^{-6})^2\,(50\,\textbf{pc})^3}{6}$

Out[•]= 2.2953×10^{50}

Characteristic Numbers in terms of energy ratios

The dimensionless numbers we use to characterize molecular clouds are often easiest to think of in terms of energy ratios. Express each of the following dimensionless numbers as a ratio of energies (possibly to a power or up to a constant coefficient):

▪ The Mach number $\mathcal{M} = \sigma_{1D}/c_s$

Pressure P is equal ϵ/C_V where $\epsilon = E_{th}/V$ is the energy density and C_V is the specific heat at constant volume for a monatomic ideal gas $=3/2$. $c_s = \sqrt{\gamma P/\rho}$ where $\rho = M/V$. Since GMC's are very efficiently cooled by CO emission, we can assume that they are isothermal and thus $\gamma = 1$. Using $E_{kin} = \frac{3}{2}\,M\,\sigma_{1D}^2$ we get

$$\mathcal{M} = \sigma_{1D}/c_s = \sqrt{\frac{E_{kin}\,2}{3\,M}} \Big/ \sqrt{P/\rho} = \sqrt{\frac{E_{kin}\,2}{3\,M}} \Big/ \sqrt{E_{th}/(C_V\,V M/V)} = \sqrt{\frac{E_{kin}}{E_{th}}}$$

- The virial ratio $\alpha_{vir} = 5\,\sigma_{1D}^2\,R/(G\,M)$

  Writing $\sigma_{1D}^2 = \frac{2}{3}\,E_{kin}/M$ and $G\,M/R$ as $\frac{5}{3}\,E_{grav}/M$ we can then write

$$\alpha_{vir} = 5\,\sigma_{1D}^2\,R/(G\,M) = 5 \times \frac{3}{3}\,E_{kin}/M\,\frac{M}{\frac{5}{3}\,E_{grav}} = \frac{2\,E_{kin}}{E_{grav}}$$

- The mass to magnetic critical mass ratio $\mu_\phi = M/M_\phi$ where M_ϕ is magnetic critical mass defined by $M_\phi = 0.12\,\phi\big/G^{1/2}$ and $\phi = \pi\,R^2\,B$ is the magnetic flux through the cloud. $B = \sqrt{6\,E_{mag}/R^3}$

$$\mu_\phi = M\big/M_\phi = \frac{M\,G^{1/2}}{0.12\,\pi\,R^2\,B} = \frac{M\,G^{1/2}}{0.12\,\pi\,R^2\,\sqrt{6\,E_{mag}/R^3}} = \frac{1}{0.12\,\pi}\sqrt{\frac{G\,M^2}{6\,R\,E_{mag}}} = 1.4\,\sqrt{\frac{E_{grav}}{E_{mag}}}$$

- The ratio of the cloud crossing time $t_{cr} = R/\sigma_{1D}$ to the free-fall time $t_{ff} = \sqrt{3\,\pi/(32\,G\,\rho)}$ where ρ is the mean density in the cloud ($\sigma_{1D} = \sqrt{2\,E_{kin}/M}$, $\rho = M\big/(\frac{4}{3}\,\pi\,R^3)$)

$$\frac{t_{cr}}{t_{ff}} = \frac{R/\sigma_{1D}}{\sqrt{3\,\pi/(32\,G\,\rho)}} = \sqrt{\frac{20}{\pi^2}\,\frac{E_{grav}}{E_{kin}}}$$

Scaling with r

Consider a spherical sub-volume of radius r inside the molecular cloud. The mass within this volume scales as $m \propto r^3$. How do the gravitational, thermal, kinetic, and magnetic energies vary with r? Assume the linewidth-size relation $\sigma_{1D} \propto l^{1/2}$

$\frac{m}{M} = \frac{r^3}{R^3}$ and $\sigma_{1D} \propto r^{1/2}$

$$E_{grav} = \frac{-3}{5}\,\frac{G\,m^2}{r} = \frac{-3}{5}\,\frac{G\,M^2}{r}\,\frac{r^6}{R^6} \propto r^5$$

$$E_{th} = \frac{3}{2}\,k_b\,T\,\frac{m}{\mu} = \frac{3}{2}\,k_b\,T\,\frac{M}{\mu}\,\frac{r^3}{R^3} \propto r^3$$

$$E_{kin} = \frac{3}{2}\,m\,\sigma_{1d}^2 = \frac{3}{2}\,M\,\frac{r^3}{R^3}\,\sigma_{1d}^2 \propto r^4$$

$$E_{magh} = \frac{B^2\,r^3}{6} \propto r^3$$

Unstable Flows

$$\frac{\partial U_i}{\partial x_i} = 0$$

$$\rho \frac{\partial U_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(U_j U_i) = \rho \nu \frac{\partial}{\partial x_j}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{\partial P}{\partial x_i}$$

The notations are time t, cartesian coordinates $(x_1, x_2, x_3) = (x, y, z)$, pressure P and velocity components $(U_1, U_2, U_3) = (U, V, W)$. The kinematic viscosity is defined as $\nu = 10^{-3}\,\mathrm{m^2/s}$, and the fluid density is $\rho = 1.0\,\mathrm{kg/m^3}$.

In[ ]:=
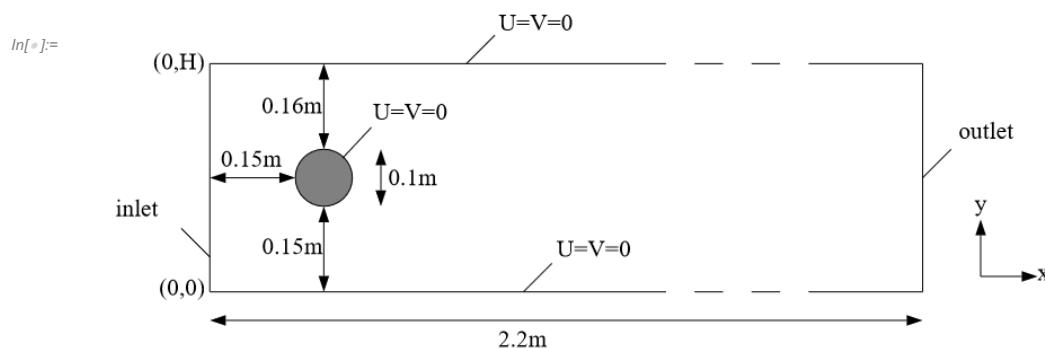
Let us consider the flow in a flat channel around a cylinder at Reynolds number = 100, when self - oscillations occur leading to the detachment of vortices in the aft part of cylinder.In this problem it is necessary to calculate drag coefficient, lift coefficient and pressure difference in the frontal and aft part of the cylinder as functions of time, maximum drag coefficient, maximum lift coefficient, Strouhal number and pressure difference ΔP (t) at t = t0 + 1/2 f.The frequency f is determined by the period of oscillations of lift coefficient f = f (c_L).The data for this test, the code and the results are shown below.

In[ ]:=
```
H = .41;
L = 2.2;
{x0, y0, r0} = {1/5, 1/5, 1/20};
Ω = RegionDifference[Rectangle[{0, 0}, {L, H}], Disk[{x0, y0}, r0]];
RegionPlot[Ω, AspectRatio → Automatic, ImageSize → Large]
```

*Out[ ]=*
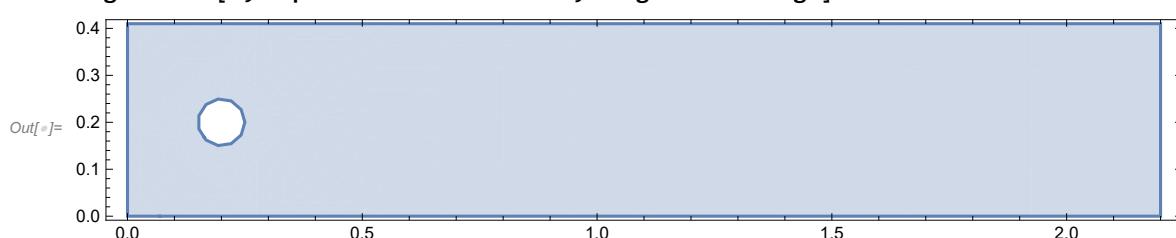
```
In[•]:= K = 400; Um = 1.5; ν = 10^-3; t0 = .02;
     U0[y_, t_] := 4 * Um * y / H * (1 - y / H) * Sin[Pi * t / 8]
     UX[0][x_, y_] := 0;
     VY[0][x_, y_] := 0;
     P0[0][x_, y_] := 0;
     Do[
       {UX[i], VY[i], P0[i]} =
       NDSolveValue[{{∇{x,y} · ({{-μ, 0}, {0, -μ}}.∇{x,y} u[x, y]) + p^(1,0) [x, y] +
              (u[x, y] - UX[i - 1][x, y]) / t0 + UX[i - 1][x, y] * D[u[x, y], x] +
              VY[i - 1][x, y] * D[u[x, y], y], ∇{x,y} · ({{-μ, 0}, {0, -μ}}.∇{x,y} v[x, y]) +
              p^(0,1) [x, y] + (v[x, y] - VY[i - 1][x, y]) / t0 +
              UX[i - 1][x, y] * D[v[x, y], x] + VY[i - 1][x, y] * D[v[x, y], y],
           u^(1,0) [x, y] + v^(0,1) [x, y]} == {0, 0, 0} /. μ → ν, {
             DirichletCondition[{u[x, y] == U0[y, i * t0], v[x, y] == 0}, x == 0.],
             DirichletCondition[{u[x, y] == 0., v[x, y] == 0.}, 0 ≤ x ≤ L && y == 0 || y == H],
             DirichletCondition[{u[x, y] == 0, v[x, y] == 0}, (x - x0)^2 + (y - y0)^2 == r0^2],
             DirichletCondition[p[x, y] == P0[i - 1][x, y], x == L]}}, {u, v, p}, {x, y} ∈ Ω,
         Method → {"FiniteElement", "InterpolationOrder" → {u -> 2, v → 2, p → 1},
           "MeshOptions" → {"MaxCellMeasure" → 0.001}}], {i, 1, K}];
```

```
In[•]:= {ContourPlot[UX[K / 2][x, y], {x, y} ∈ Ω, AspectRatio → Automatic,
       ColorFunction → "BlueGreenYellow", FrameLabel → {x, y}, PlotLegends → Automatic,
       Contours → 20, PlotPoints → 25, PlotLabel → u, MaxRecursion → 2],
      ContourPlot[VY[K / 2][x, y], {x, y} ∈ Ω, AspectRatio → Automatic,
       ColorFunction → "BlueGreenYellow", FrameLabel → {x, y},
       PlotLegends → Automatic, Contours → 20, PlotPoints → 25,
       PlotLabel → v, MaxRecursion → 2, PlotRange → All]} // Quiet
```
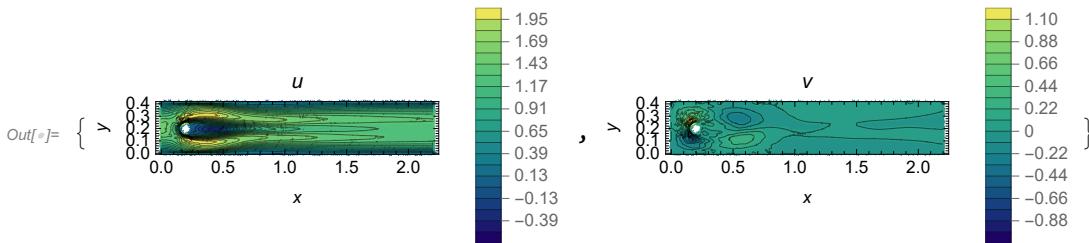


```
In[•]:= {DensityPlot[UX[K][x, y], {x, y} ∈ Ω, AspectRatio → Automatic,
       ColorFunction → "BlueGreenYellow", FrameLabel → {x, y},
       PlotLegends → Automatic, PlotPoints → 25, PlotLabel → u, MaxRecursion → 2],
      DensityPlot[VY[K][x, y], {x, y} ∈ Ω, AspectRatio → Automatic,
       ColorFunction → "BlueGreenYellow", FrameLabel → {x, y}, PlotLegends → Automatic,
       PlotPoints → 25, PlotLabel → v, MaxRecursion → 2, PlotRange → All]} // Quiet
```
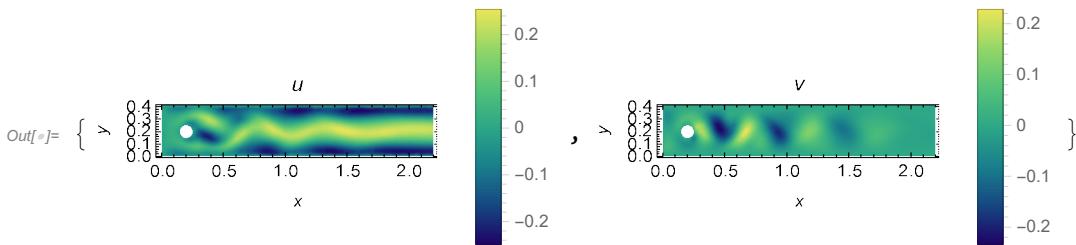
```
In[ ]:= anim = Table[Rasterize[DensityPlot[UX[K][x, y], {x, y} ∈ Ω, AspectRatio → Automatic,
        ColorFunction → "BlueGreenYellow", FrameLabel → {x, y}, PlotLegends → Automatic,
        PlotPoints → 25, PlotLabel → u, MaxRecursion → 2, ImageSize → Large]], {K, 1, 400}];
```

```
In[ ]:= ListAnimate[anim]
```

Out[ ]=