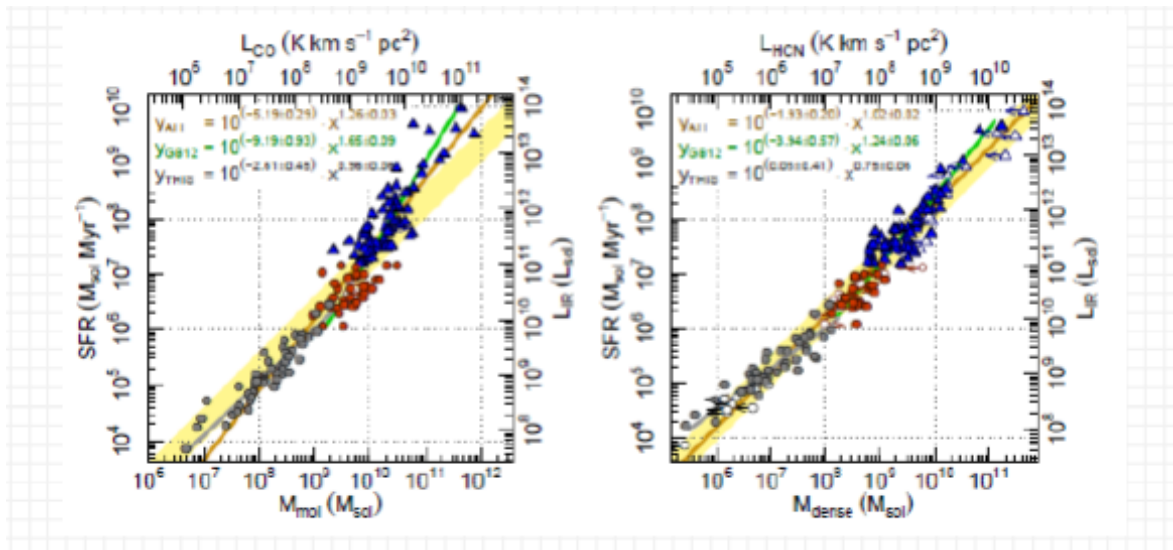


# Q&A Session 02.06.2020

## Star formation rate on galactic scales

### Physical Interpretation and Depletion Times



Can we convert the HCN emission into gas mass?

This requires an "HCN X-Factor" Since the HCN J = 1 - 0 line (and other low J lines) is generally optically thick, such a conversion factor can be derived from theoretical arguments much like the ones we used to estimate  $X_{\text{CO}}$ .

The conversion factor does not depend on the HCN abundance, which is good, because that is not tremendously well known.

However, the resulting conversion is still significantly more uncertain than for CO, because, unlike the case for CO, it has not been calibrated against independent tracers of the mass like dust or  $\gamma$ -rays.

HCN luminosity  $\rightarrow$  mass

observed IR luminosity  $\rightarrow$  star formation rate

$\rightarrow$  depletion time for the HCN-emitting gas.

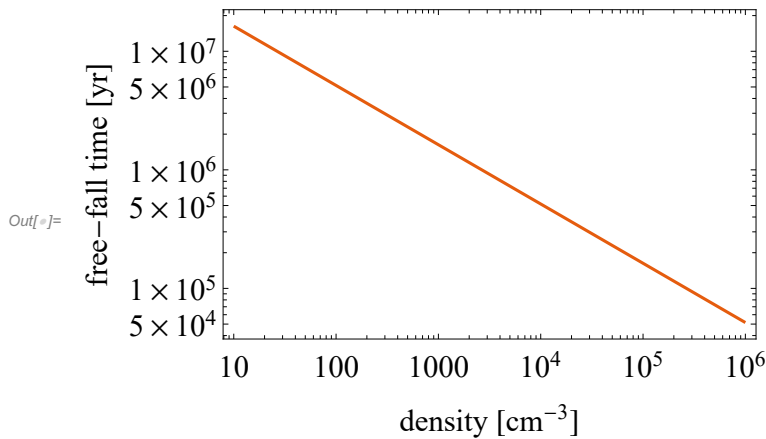
Typical depletion times are  $\sim 10 - 100$  Myr (much smaller than for CO) But remember, we are looking at much denser gas! Assuming a reasonable density we can estimate the free-fall time:

$$\ln[*]:= \text{fftime}[n_] := \sqrt{\frac{3 \pi}{32 \times 6.6726 \times 10^{-8} \times 1.67262 \times 10^{-24} n}}$$

$$\text{seconds2years} = (365 \times 24 \times 3600)^{-1};$$

```
In[*]:= fftime[105] seconds2years
```

```
Out[*]:= 162895.
```



So we find a free-fall time of about  $10^5$  yrs at a density of  $\sim 10^5 \text{ cm}^{-3}$ . Assuming a depletion time of  $t_{\text{dep}} = 10 \text{ Myr}$  we find a star formation efficiency of:

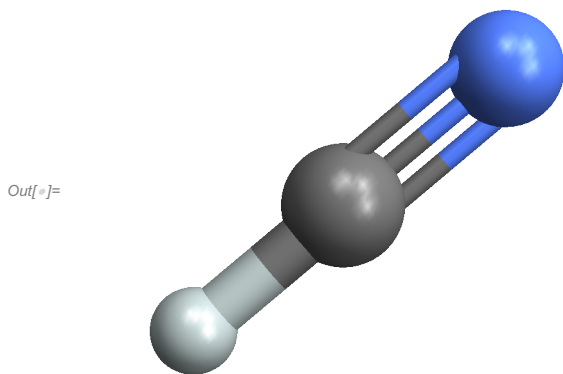
$$\epsilon_{\text{ff}} = \frac{t_{\text{ff}}}{t_{\text{dep}}} \sim 10^{-3} \dots 10^{-2}$$

The uncertainty is pretty large, but a  $\epsilon_{\text{ff}} \sim 1$  is basically ruled out.

## Radiative transfer of HCN

### a) Rotational transitions of HCN

Consider the rotational transitions of the HCN molecule, i.e. the transitions  $\Delta J = J_{\text{up}} - J_{\text{low}} = 1$  with the angular momentum quantum number  $J$ .



see also KIDA

Determine the transition frequencies (in GHz) and the level energies (in Kelvin) of the first 20 levels above the ground state. The rotational constant  $B$  is 44316 MHz. Assume HCN to be a simple, stiff,

linear rotator.

Remark: Write a small computer program to perform the calculations.

## Solution

The energy of a rotational level is defined as:

$$E_{J,0} = B J(J+1) h \quad (1)$$

this is the energy difference between J and J=0.

```
In[1]:= {c -> 2.99792458 × 1010, h -> 6.62608 × 10-27, k -> 1.38066 × 10-16};
```

```
In[2]:= EnergyLevel[J_, B_] := B J (J + 1) 6.62608 × 10-27
```

Since  $E_{J,0} = h \nu$  it follows that  $h \nu_{J,0} = B J(J+1) h$ .

$$h \nu_{J,0} = B J(J+1) h \quad (2)$$

hence, the frequency difference between the levels J and J-1 is

$$\nu_{J,J-1} = \nu_{J,0} - \nu_{J-1,0} = B J(J+1) - (B (J-1)((J-1)+1)) = B J((J+1) - (J-1)) = 2 B J \quad (3)$$

```
In[3]:= TransitionFrequency[Jupper_, B_] := 2 B Jupper
```

The statistical weight of a J-level are defined as  $g_J = 2 J + 1$

```
In[4]:= g[J_] := 2 J + 1
```

```
In[5]:= tab =
```

```
Table[{ToString[j] <> "→" <> ToString[j - 1], TransitionFrequency[j, 44316. × 106] 10-9,
EnergyLevel[j, 44316 × 106] / (1.38 × 10-16), 2 j + 1, 2 (j - 1) + 1}, {j, 1, 20}];
```

```
In[6]:= TableForm[tab[[All, 2 ;; -1]],
TableHeadings → {tab[[All, 1]], {"νJ,J-1 [GHz]", "EJ,0[K]", "gu", "gl"}}]
```

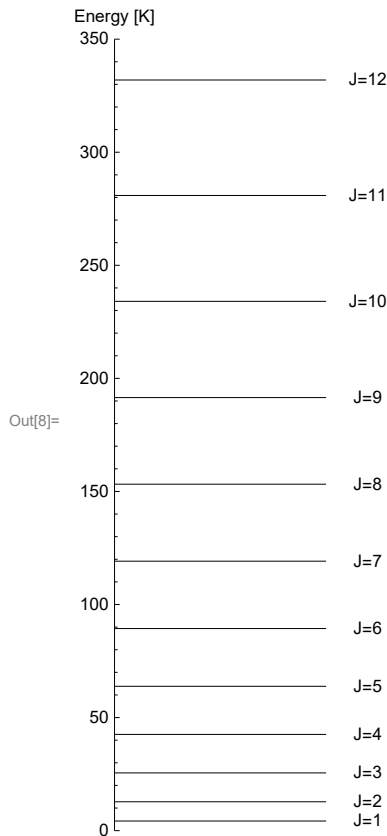
Out[6]//TableForm=

	$\nu_{J,J-1}$ [GHz]	$E_{J,0}$ [K]	$g_u$	$g_l$
1→0	88.632	4.25567	3	1
2→1	177.264	12.767	5	3
3→2	265.896	25.534	7	5
4→3	354.528	42.5567	9	7
5→4	443.16	63.8351	11	9
6→5	531.792	89.3691	13	11
7→6	620.424	119.159	15	13
8→7	709.056	153.204	17	15
9→8	797.688	191.505	19	17
10→9	886.32	234.062	21	19
11→10	974.952	280.874	23	21
12→11	1063.58	331.942	25	23
13→12	1152.22	387.266	27	25
14→13	1240.85	446.846	29	27
15→14	1329.48	510.681	31	29
16→15	1418.11	578.771	33	31
17→16	1506.74	651.118	35	33
18→17	1595.38	727.72	37	35
19→18	1684.01	808.578	39	37
20→19	1772.64	893.691	41	39

We now have all information to create an energy level diagram:

```
In[7]:= rotLevel[J_, B_] :=
  Graphics[{Line[{{0, #}, {1, #}}], Text[Style["J=" <> ToString[J]], {1.2, #}]} &@
  (EnergyLevel[J, B] / (1.38 × 10-16)), AspectRatio → 3/1]
```

```
In[8]:= Show[Table[rotLevel[j, 44 316 × 106], {j, 1, 12}], Axes → {False, True},
  PlotRange → {0, 350}, Ticks → {None, Automatic}, AxesLabel → {None, "Energy [K]"}]
```



## b) Level occupation

Assuming LTE (Local Thermal Equilibrium), write a small program to calculate the level occupation numbers of the rotational levels relative to the ground state ( $n_0 = 1$ ). Assume excitation temperatures  $T_{\text{ex}} = 35$  K, 50 K, and 100 K.

Remember that the levels are populated according to a Boltzmann distribution.

### Solution

Level occupation under LTE conditions is defined according to the Boltzmann distribution

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} \exp\left(-\frac{E_u - E_l}{T_{\text{ex}}}\right) \quad (4)$$

```
In[9]:= populationRatio[J_, T_, B_] :=  $\frac{g[J]}{g[J - 1]} \text{Exp}\left[-\frac{\text{EnergyLevel}[J, B] - \text{EnergyLevel}[J - 1, B]}{(1.38 \times 10^{-16}) T}\right]$ 
```

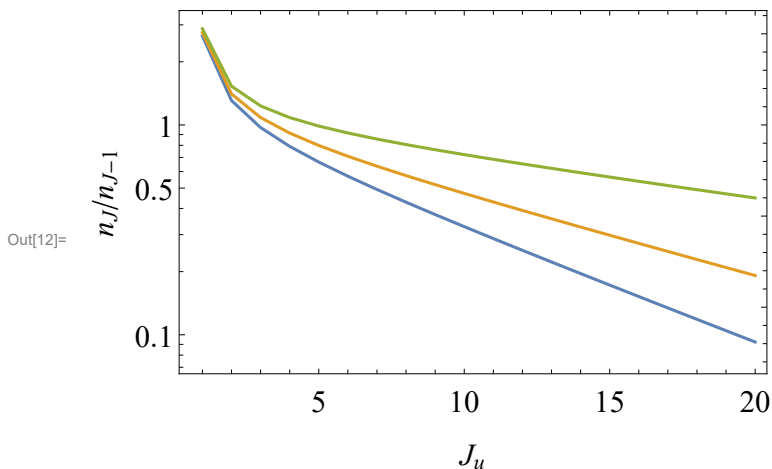
```
In[10]:= population35K = Table[populationRatio[J, 35, 44 316 × 106], {J, 1, 20}]
```

```
Out[10]= {2.65653, 1.30688, 0.972097, 0.790533, 0.665456, 0.569789,
  0.492612, 0.428458, 0.374153, 0.327646, 0.287502, 0.252661, 0.222302,
  0.195771, 0.172533, 0.152144, 0.13423, 0.118474, 0.104604, 0.0923839}
```

```
In[11]:= population = Table[populationRatio[J, T, 44 316 × 106], {T, {35, 50, 100}}, {J, 1, 20}]
```

```
Out[11]= {{2.65653, 1.30688, 0.972097, 0.790533, 0.665456, 0.569789,
0.492612, 0.428458, 0.374153, 0.327646, 0.287502, 0.252661, 0.222302,
0.195771, 0.172533, 0.152144, 0.13423, 0.118474, 0.104604, 0.0923839},
{2.75522, 1.40579, 1.08451, 0.914718, 0.798599, 0.709194, 0.635913,
0.573645, 0.519548, 0.47187, 0.429439, 0.391418, 0.357181, 0.326238,
0.298194, 0.272724, 0.249552, 0.228442, 0.20919, 0.191617},
{2.87501, 1.53068, 1.2322, 1.08447, 0.98796, 0.915499, 0.85659, 0.806307,
0.762018, 0.722178, 0.685812, 0.652268, 0.621092, 0.591949,
0.564588, 0.538813, 0.514467, 0.491422, 0.469572, 0.448824}}
```

```
In[12]:= ListLogPlot[population, Joined → True, Frame → True,
Axes → False, FrameLabel → {"Ju", "nJ/nJ-1"}, PlotRange → Full,
BaseStyle → {FontSize → 16, FontFamily → "Times"}]
```



### c) Partition function

The partition function of linear molecules can be approximated as  $Z \approx k T_{\text{ex}} / (h B)$ . Write a small computer program to perform the partition function explicitly and determine how many levels you have to include in order to make the deviation between the explicit sum and the approximation < 1% (for  $T_{\text{ex}} = 35 \text{ K}$ ,  $50 \text{ K}$ , and  $100 \text{ K}$ ).

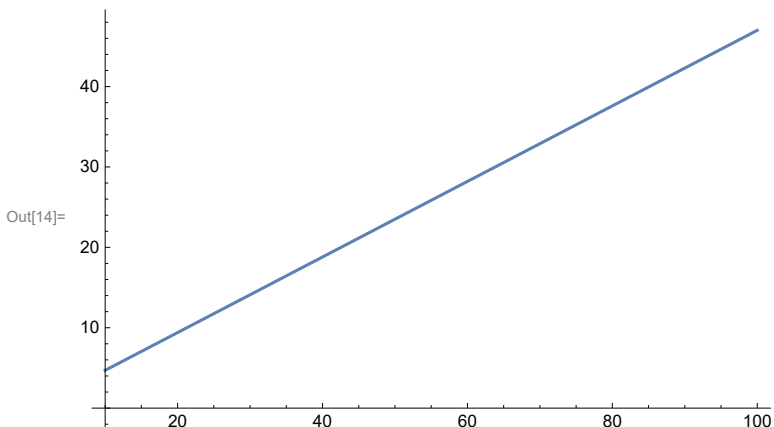
Remember to account for the statistical weights  $g_J$  in the explicit summation.

### Solution

```
In[13]:= Zapprox[T_, B_] := 
$$\frac{1.38 \times 10^{-16} \text{ T}}{6.62608 \times 10^{-27} \text{ B}}$$

```

In[14]:= Plot[Zapprox[T, 44 316 × 10<sup>6</sup>], {T, 10, 100}]



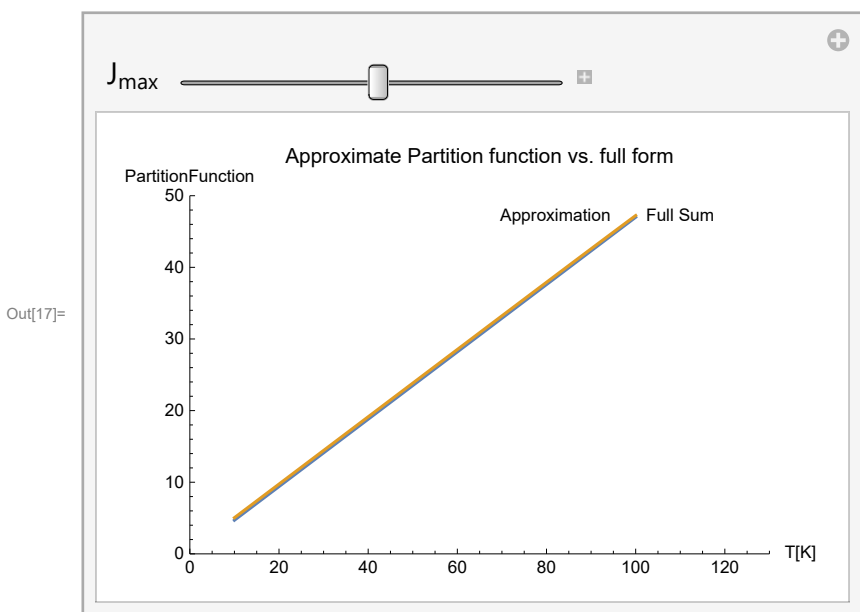
This is the full definition of the partition function:

In[15]:= ZFull[T\_, B\_, Jmax\_] := Sum[g[j] Exp[-EnergyLevel[j, B] / (1.38 × 10<sup>-16</sup> T)], {j, 0, Jmax}]

For example for the levels J=0-3:

In[16]:= ZFull[T, 44 316 × 10<sup>6</sup>, 3]

Out[16]= 1. + 7 e<sup>-25.534/T</sup> + 5 e<sup>-12.767/T</sup> + 3 e<sup>-4.25567/T</sup>



Now we can compare the approximation with the full form and determine when they both agree within 1%:

```
In[ ]:= Labeled[TableForm[Table[{Style[NumberForm[#, {3, 2}], If[Abs[#] < 1., Red, Black]] &[
  100  $\frac{\text{Zapprox}[T, 44\,316 \times 10^6] - \text{ZFull}[T, 44\,316 \times 10^6, j]}{\text{ZFull}[35, 44\,316 \times 10^6, j]}$  ]],
  {j, 1, 20}, {T, {35, 50, 100}}], TableHeadings ->
  {Automatic, {"T=35K", "T=50K", "T=100K"}}, "Error in %", Top]
```

... **NumberForm**: Requested number precision is lower than number of digits shown; padding with zeros.

		Error in %		
		T=35K	T=50K	T=100K
	1	350.00	540.00	1180.00
	2	131.00	223.00	543.00
	3	56.60	111.00	317.00
	4	24.90	59.40	208.00
	5	10.00	31.80	145.00
	6	3.07	16.20	102.00
	7	-0.05	7.28	71.30
	8	-1.33	2.42	48.30
	9	-1.80	-0.05	31.40
Out[ ]:=	10	-1.95	-1.21	19.30
	11	-2.00	-1.71	11.00
	12	-2.01	-1.90	5.64
	13	-2.01	-1.97	2.30
	14	-2.01	-1.99	0.32
	15	-2.01	-2.00	-0.80
	16	-2.01	-2.00	-1.41
	17	-2.01	-2.00	-1.72
	18	-2.01	-2.00	-1.87
	19	-2.01	-2.00	-1.94
	20	-2.01	-2.00	-1.97

#### d) Detection equation and excitation temperature(4 points)

Spectral lines of several HCN isotopomers are observed in two astronomical sources. The measurements are summarized in the following table:

spectral line	line intensity
	$T_b$ (K)
source A	
HCN $J = 1 - 0$	16.1
$H^{13}CN$ $J = 1 - 0$	0.32
source B	
HCN $J = 1 - 0$	26.3
$H^{13}CN$ $J = 1 - 0$	8.6

Assume a Gaussian spectral shape of the lines and an elemental ratio of  $HCN/H^{13}CN=50$ .

Estimate without extensive calculation which of the above lines are optically thin. Using the detection equation:

$$T_b = (1 - e^{-\tau_\nu}) (J_\nu(T_{ex}) - J_\nu(T_{bg})) \quad (5)$$

and

$$J_{\nu}(T) = \frac{h \nu}{k \left( e^{\frac{h \nu}{k T}} - 1 \right)}, \quad (6)$$

the equations for HCN line ratio can be written as:

$$\frac{T_b^{12}}{T_b^{13}} = \frac{(1 - e^{-\tau_{v,12}}) (J_{\nu}(T_{\text{ex}}^{\text{HCN}}) - J_{\nu}(T_{\text{bg}}))}{(1 - e^{-\tau_{v,13}}) (J_{\nu}(T_{\text{ex}}^{\text{H}^{13}\text{CN}}) - J_{\nu}(T_{\text{bg}}))} \quad (7)$$

Use equation (7) to derive the excitation temperature of the line. This is possible for only one of the two sources. Explain why. Assume  $T_{\text{ex}}(\text{HCN}) = T_{\text{ex}}(\text{H}^{13}\text{CN})$ ,  $\nu_{12} = \nu_{13}$ , and  $T_{\text{bg}} = 0 \text{ K}$ .

## Solution

Which lines are optically thin?

- If both lines are optically thin (and assuming LTE)  $\frac{T_b^{12}}{T_b^{13}} = \text{elemental ratio}$
- If both lines are optically thick (and assuming LTE)  $\frac{T_b^{12}}{T_b^{13}} = \frac{T_{\text{ex}}^{12}}{T_{\text{ex}}^{13}} \approx 1$
- If  $1 \leq \frac{T_b^{12}}{T_b^{13}} \leq \text{elemental ratio}$  then  $T_b^{12}$  is optically thick and  $T_b^{13}$  is optically thin

Source A:  $\frac{T_b^{12}}{T_b^{13}}$

**16.1**

**0.13**

123.846

123 >>  $\frac{\text{HCN}}{\text{H}^{13}\text{CN}} = 50$

This ratio is larger than the elemental ratio and we have to assume that, e.g. that cloud is too small to host a sufficient column of  $\text{H}^{13}\text{CN}$ , and that both transitions are still optically thin.

Source B:  $\frac{T_b^{12}}{T_b^{13}}$

**26.3**

**8.6**

3.05814

The ratio of  $\sim 3$  indicates that the  $\text{H}^{12}\text{CN}$  line is optically thick and  $\text{H}^{13}\text{CN}$  line is optically thin.

We now make the following assumptions and simplifications:

1.  $T_{\text{ex}}(\text{HCN}) = T_{\text{ex}}(\text{H}^{13}\text{CN})$
2.  $\nu_{12} = \nu_{13}$
3.  $J_{\nu_{12}}(T_{12}) = J_{\nu_{13}}(T_{13})$
4.  $T_{\text{bg}} = 0 \text{ K}$

This gives

$$\frac{T_b^{12}}{T_b^{13}} = \frac{(1 - e^{-\tau_{v,12}}) (J_{\nu}(T_{\text{ex}}^{\text{HCN}}) - J_{\nu}(T_{\text{bg}}))}{(1 - e^{-\tau_{v,13}}) (J_{\nu}(T_{\text{ex}}^{\text{H}^{13}\text{CN}}) - J_{\nu}(T_{\text{bg}}))} = \frac{(1 - e^{-\tau_{v,12}})}{(1 - e^{-\tau_{v,13}})} \quad (8)$$

assuming both transitions are optically thin:



$$\frac{T_b^{12}}{T_b^{13}} = \frac{\tau_{\text{HCN}}}{\tau_{H^{13}\text{CN}}} = \frac{50}{1} \quad (9)$$

but since under optically thin conditions  $\tau_{\text{HCN}}=50 \tau_{H^{13}\text{CN}}$ , equation (9) has no unique solution.

If  $H^{12}\text{CN}$  is optically thick and  $H^{13}\text{CN}$  is optically thin:

$$\frac{T_b^{12}}{T_b^{13}} = \frac{(1 - e^{-\tau_{v,12}})}{(1 - e^{-\tau_{v,13}})} = \frac{1}{\tau_{H^{13}\text{CN}}} = \frac{1}{\alpha \tau_{\text{HCN}}} \quad (10)$$

with  $\alpha = \frac{1}{50}$ . Ergo:

$$3 = \frac{1}{\frac{1}{50} \tau_{\text{HCN}}} \Rightarrow \tau_{\text{HCN}} = \frac{50}{3} \sim 16.7 \quad (11)$$

We now can derive the excitation temperature:

$$T_b^{12} = (1 - e^{-\tau_{v,12}}) J_\nu(T_{\text{ex}}^{\text{HCN}}) \quad (12)$$

$$(1 - e^{-\tau_{v,12}}) \approx 1 \quad (13)$$

$$T_b^{12} = \frac{h\nu}{k} \frac{1}{\exp\left(\frac{h\nu}{kT_{\text{ex}}}\right) - 1} \quad (14)$$

and solving for  $T_{\text{ex}}$

$$T_{\text{ex}} = \frac{h\nu}{k} \frac{1}{\ln\left(\frac{h\nu}{kT_b^{12}} + 1\right)} \quad (15)$$

$$\text{HCN } J=1 \rightarrow 0 \Rightarrow \frac{h\nu}{k} = 4.3 \text{ K}$$

$$\text{In[*]:= } T_{\text{ex}} = \frac{4.3}{\text{Log}\left[\frac{4.3}{26.3} + 1\right]}$$

$$\text{Out[*]:= } 28.3958$$

## Inferring Star Formation Rates in the Infrared.

One way to measure the star formation rate of a galaxy is to measure the galaxy's infrared luminosity. The underlying assumptions behind this method are that (1) most of the total radiant output in the galaxy comes from young, recently formed stars, and (2) that in a sufficiently dusty galaxy most of the starlight will be absorbed by dust grains within the galaxy and then re-radiated in the infrared.

Use Starburst99 with the default parameters to compute the total luminosity of a stellar population in which star formation occurs continuously at a fixed rate  $\dot{M}_*$ .

What is the ratio of  $L_{\text{tot}}/\dot{M}_*$  after 10 Myr? After 100 Myr? After 1 Gyr?

Compare these ratios to the conversion factor between  $L_{\text{IR}}$  and  $\dot{M}_*$  given in Kennicutt (1998, ARA&A, 36, 189).

We chose a fixed SFR  $\dot{M}_*=1 M_\odot \text{ yr}^{-1}$  and assume two different IMFs:

I) Kroupa IMF  $\alpha = (1.3; 2.3)$  with limits  $M = (0.1; 0.5; 100)$

II) top-heavy IMF  $\alpha = 2.3$  with limits  $M = (0.5; 100)$

Both simulations run for  $t = 1$  Gyr. We convert the bolometric absolute magnitudes to total luminosities and find for IMF I):

$$t = 10 \text{ Myr} \Rightarrow \frac{L_{\text{tot}}}{\dot{M}_*} = 2.34 \times 10^{43} \frac{\text{erg s}^{-1}}{M_{\odot} \text{ yr}^{-1}}$$

$$t = 100 \text{ Myr} \Rightarrow \frac{L_{\text{tot}}}{\dot{M}_*} = 3.55 \times 10^{43} \frac{\text{erg s}^{-1}}{M_{\odot} \text{ yr}^{-1}}$$

$$t = 1 \text{ Gyr} \Rightarrow \frac{L_{\text{tot}}}{\dot{M}_*} = 4.57 \times 10^{43} \frac{\text{erg s}^{-1}}{M_{\odot} \text{ yr}^{-1}}$$

Kennicutt found in his 1998 paper:

$$\text{SFR} (M_{\odot} \text{ yr}^{-1}) = 4.5 \times 10^{-44} L_{\text{FIR}} (\text{erg s}^{-1})$$

therefore:

$$= \frac{L_{\text{FIR}} (\text{erg s}^{-1})}{\text{SFR} (M_{\odot} \text{ yr}^{-1})} = 2.22222 \times 10^{43}$$

The important assumption here of  $L_{\text{FIR}} \sim L_{\text{tot}} \sim \text{SFR}$  is only reasonable if the IR luminosity is mostly coming from the ISM and NOT from the stars. If the number of late-type stars increases significantly, this assumption breaks down. This explains our discrepancy for later times!

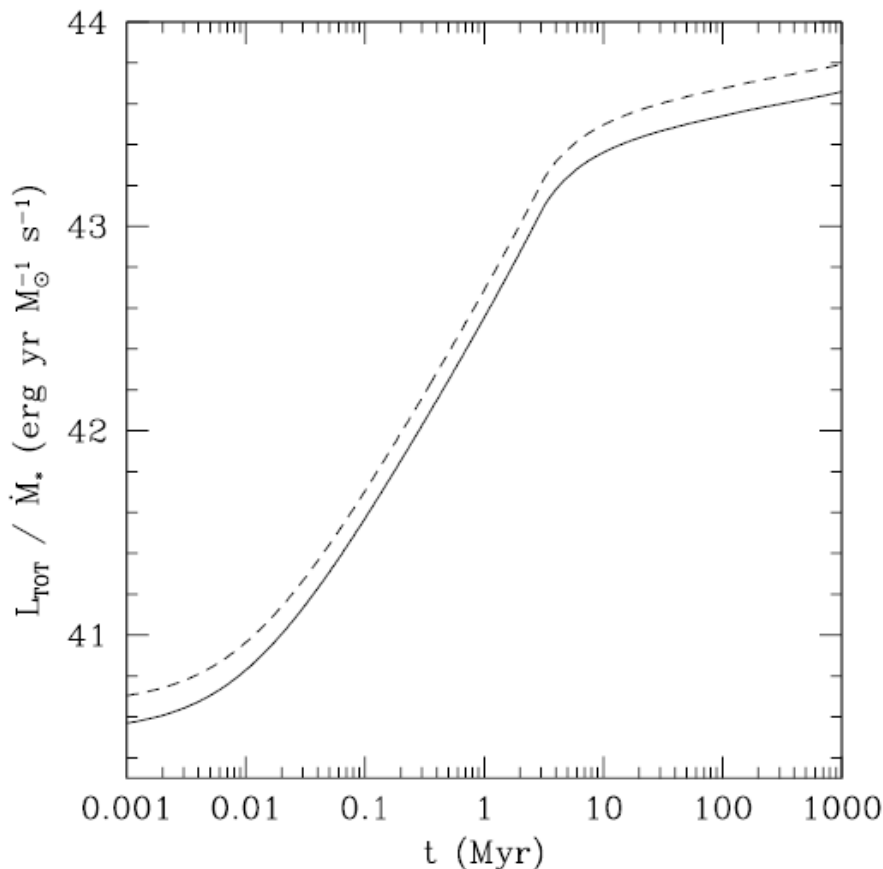


Figure 1:  $\frac{L_{\text{tot}}}{\dot{M}_*}$  as a function of the time. Dashed line is for top heavy IMF. Solid line is for Kroupa IMF.

Based on this plot, how old does a stellar population have to be before  $L_{\text{IR}}$  becomes a good tracer of

the total star formation rate?

It seems, that after  $t \sim 1 - 5$  Myr the bolometric luminosity traces the SFR in a stable way. This seems to correspond to the time required for early type stars to reach the equilibrium between formation and explosion via SN.

For a top-heavy IMF slightly: How much does the luminosity change for a fixed star formation rate? What do you infer from this about how sensitive this technique is to assumptions about the form of the IMF?

Looking at the dashed line in Fig 1 we estimate a difference to simulation I of  $\sim 36\%$ . This demonstrates how sensitive this technique is to the assumed (real) IMF.

## Molecular Cloud Masses and Toomre Instability.

The molecular cloud mass function appears to be well-described by a truncated power-law. This is interesting because the truncation implies the existence of a characteristic mass scale. GMCs probably form via gravitational instability in a galactic disk, so this process is a natural place to look for the origin of that scale.

a)

Consider a uniform infinitely thin disk of surface density  $\Sigma$  occupying the  $z = 0$  plane. The disk is rotating at a constant angular speed  $\vec{\Omega} = \Omega \vec{e}_z$ . The velocity of the fluid in the  $z = 0$  plane is  $\vec{v}$  and its vertically-integrated pressure is  $\Pi = \int_{-\infty}^{+\infty} P dz = \Sigma c_s^2$ . Consider a coordinate system co-rotating with the disk. In this frame, we can derive the governing equations by combining the vertically-integrated equations of motion with the Poisson equation. This gives

$$\begin{aligned} \frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \vec{v}) &= 0 \\ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} &= -\frac{\nabla \Pi}{\Sigma} - \nabla \phi - 2 \Omega \times \vec{v} + \Omega^2 (x \vec{e}_x + y \vec{e}_y) \\ \nabla^2 \phi &= 4 \pi G \Sigma \delta(z) \end{aligned}$$

The last two terms in the second equation are the Coriolis and centrifugal force terms. We wish to perform a stability analysis of these equations.

Consider a solution  $(\Sigma_0, \phi_0)$  to these equations in which the gas is in equilibrium (i.e.  $v = 0$ ), and add a small perturbation:  $\Sigma = \Sigma_0 + \epsilon \Sigma_1$ ,  $v = v_0 + \epsilon v_1$ ,  $\phi = \phi_0 + \epsilon \phi_1$ , where  $\epsilon \ll 1$ . Derive the perturbed equations by substituting these values of  $\Sigma$ ,  $v$ , and  $\phi$  into the equations of motion and keeping all the terms that are linear in  $\epsilon$ .

Substituting in the perturbed terms for  $\Sigma$ ,  $v$ , and  $\phi$ , the linearized equation of mass conservation is

$$\begin{aligned} \frac{\partial}{\partial t} (\Sigma_0 + \epsilon \Sigma_1) + \nabla \cdot [ (\Sigma_0 + \epsilon \Sigma_1) (v_0 + \epsilon v_1) ] &= 0 \\ \frac{\partial}{\partial t} \Sigma_1 + \Sigma_0 \nabla \cdot v_1 + \nabla \cdot (\Sigma_1 v_0) &= 0 \end{aligned}$$

In going from the first line to the second, we dropped terms of order  $\epsilon^2$ , we used the fact that  $\Sigma_0$  is constant in time to drop the term  $\partial \Sigma_0 / \partial t$ , and we used the fact that it is constant in space (since the

unperturbed state is uniform) to take the  $\Sigma_0$  factor out of the divergence. Note that  $v_0$  and  $\Sigma_1$  are not constant in space, so they cannot be taken out of the divergence.

The linearized momentum equation is

$$\frac{\partial}{\partial \mathbf{t}} (\mathbf{v}_\theta + \epsilon \mathbf{v}_1) + (\mathbf{v}_\theta + \epsilon \mathbf{v}_1) \nabla (\mathbf{v}_\theta + \epsilon \mathbf{v}_1) = - \frac{\nabla (\Sigma_\theta + \epsilon \Sigma_1)}{(\Sigma_\theta + \epsilon \Sigma_1)} c_s^2 - \nabla (\phi_\theta + \epsilon \phi_1) - 2 \Omega \times (\mathbf{v}_\theta + \epsilon \mathbf{v}_1) + \Omega^2 (\mathbf{x} \vec{e}_x + \mathbf{y} \vec{e}_y)$$

To simplify this, we recall that, since the equilibrium is an exact solution, it must be the case that

$$\frac{\partial}{\partial \mathbf{t}} \mathbf{v}_\theta + \mathbf{v}_\theta \nabla \mathbf{v}_\theta = - \frac{\nabla \Sigma_\theta}{\Sigma_\theta} c_s^2 - \nabla \phi_\theta - 2 \Omega \times \mathbf{v}_\theta + \Omega^2 (\mathbf{x} \vec{e}_x + \mathbf{y} \vec{e}_y)$$

and we can therefore cancel these terms. Doing so, and dropping terms of order  $\epsilon^2$ , we are left with

$$\frac{\partial}{\partial \mathbf{t}} \mathbf{v}_1 + \mathbf{v}_\theta \nabla \mathbf{v}_1 + \mathbf{v}_1 \nabla \mathbf{v}_\theta = - \frac{\nabla \Sigma_1}{\Sigma_1} c_s^2 - \nabla \phi_1 - 2 \Omega \times \mathbf{v}_1$$

Finally, the linearized Poisson equation is

$$\nabla^2 (\phi_\theta + \epsilon \phi_1) = 4 \pi G (\Sigma_\theta + \epsilon \Sigma_1) \delta(\mathbf{z})$$

$$\nabla^2 \phi_1 = 4 \pi G \Sigma_1 \delta(\mathbf{z})$$

In deriving the second line we used the fact that the unperturbed state is an exact solution to cancel  $\nabla^2 \phi_0$  with  $4 \pi G \Sigma_0 \delta(\mathbf{z})$ .

b)

The perturbed equations can be solved by Fourier analysis. Consider a trial value of  $\Sigma_1$  described by a single Fourier mode  $\Sigma_1 = \Sigma_a \exp[i(kx - \omega t)]$ , where we choose to orient our coordinate system so that the wave vector  $k$  for this mode is in the  $x$  direction. As an ansatz for  $\phi_1$ , we will look for a solution of the form  $\phi_1 = \phi_a \exp[i(kx - \omega t) - |kz|]$ . (One can show that the solution must take this form, but we will not do so here.) Derive the relationship between  $\phi_a$  and  $\Sigma_a$ .

First, we plug the Fourier mode trial solutions into the Poisson equation:

$$\nabla^2 \phi_a \exp[i(kx - \omega t) - |kz|] = 4 \pi G \Sigma_a \exp[i(kx - \omega t)] \delta(\mathbf{z})$$

To eliminate the  $\delta(\mathbf{z})$ , we now integrate both sides in  $z$  over a range  $[-\zeta, \zeta]$  and evaluate in the limit  $\zeta \rightarrow 0$ . This gives

$$\begin{aligned} \phi_a \int_{-\zeta}^{\zeta} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \exp[i(kx - \omega t) - |kz|] dz \\ = 4 \pi G \Sigma_a \exp[i(kx - \omega t)] \int_{-\zeta}^{\zeta} \delta(\mathbf{z}) dz \\ = 4 \pi G \Sigma_a \exp[i(kx - \omega t)] \end{aligned}$$

To evaluate the left-hand side, note that the  $\partial^2 / \partial y^2$  term vanishes because there is no  $y$ -dependence, and the  $\partial^2 / \partial x^2$  term will also vanish when we take the limit  $\zeta \rightarrow 0$ , because the integrand is finite. Only the  $\partial^2 / \partial z^2$  term will survive. Thus we have

$$4 \pi G \Sigma_a = \phi_a \lim_{\zeta \rightarrow 0} \int_{-\zeta}^{\zeta} \frac{\partial^2}{\partial z^2} \exp[-|kz|] dz$$

$$= \phi_a \lim_{\zeta \rightarrow 0} \left[ \left( \frac{d}{dz} \exp[-|kz|] \right)_{z=\zeta} - \left( \frac{d}{dz} \exp[-|kz|] \right)_{z=-\zeta} \right] dz$$

$$= -2 \phi_a |k|$$

Thus we have

$$\phi_a = \frac{2 \pi G \Sigma_a}{|k|}$$

c)

Now try a similar single-Fourier mode form for the perturbed velocity:

$v_1 = (v_{ax} \vec{e}_x + v_{ay} \vec{e}_y) \exp[i(kx - \omega t)]$ . Derive three equations relating the unknowns  $\Sigma_a$ ,  $v_{ax}$ , and  $v_{ay}$ . You will find it useful to expand  $\Omega$  in a Taylor series around the origin of your coordinate system, i.e., write  $\Omega = \Omega_0 + (d\Omega/dx)_0 x$ , where  $\Omega_0 = v_R/R$  and  $(d\Omega/dx)_0 = -\Omega_0/R$ .

As a first step, let us rewrite the terms involving  $v_0$  in a more convenient form; this is the Taylor expansion part. Recall that we are in a frame that is co-rotating with the disk, and where  $x$  is the distance from the center of our co-rotating reference frame in the radial direction. In the lab frame, the velocity is  $\dot{v}_0 = v_R \vec{e}_\phi$ , and the velocity of the co-rotating reference frame at a distance  $r$  from the origin is  $v_{rot} = \Omega_0 r \vec{e}_\phi$ . The unperturbed velocity in the rotating frame is the difference between these two, i.e.,

$$\begin{aligned} v_\theta &= v_\theta' - v_{rot} \\ &= (v_R - \Omega_0 r) \vec{e}_y \\ &= (\Omega_0 R - \Omega_0 (R + x)) \vec{e}_\phi = -\Omega_0 x \vec{e}_y \end{aligned}$$

where we have used the fact that  $\vec{e}_\phi$  in the lab frame is the same as  $\vec{e}_y$  in our co-rotating frame.

With this result in hand, we can now begin to make substitutions into the perturbed equations. The perturbed equation of mass conservation becomes

$$-i\omega \Sigma_a + ik \Sigma_a v_{ax} = 0$$

The momentum equation becomes

$$-i\omega (v_{ax} \vec{e}_x + v_{ay} \vec{e}_y) - \Omega_0 v_{ax} + ik \Sigma_a v_{ax} \vec{e}_y = -ik \frac{\Sigma_a}{\Sigma_\theta} c_s^2 \vec{e}_x - ik \phi_\theta \vec{e}_x - 2 \vec{\Omega} \times (v_{ax} \vec{e}_x + v_{ay} \vec{e}_y)$$

Since  $\vec{\Omega} = \Omega \vec{e}_z$  we can write out the two components of this equation as

$$\begin{aligned} -i\omega v_{ax} &= -ik \frac{\Sigma_a}{\Sigma_\theta} c_s^2 \vec{e}_x + ik \frac{2 \pi G \Sigma_a}{|k|} + 2 \Omega_0 v_{ay} \\ -i\omega v_{ay} &= \Omega_0 v_{ax} \end{aligned}$$

where we have evaluated the equation at  $x = 0$  and thus we have  $\Omega = \Omega_0$ , and in the first equation we have substituted in for  $\phi_\theta$ . We now have three equations in the three unknowns  $\Sigma_a$ ,  $v_{ax}$ , and  $v_{ay}$ .

d)

Show that these equations have non-trivial solutions only if

$$\omega^2 = 2 \Omega_0^2 - 2 \pi G \Sigma_0 |k| + k^2 c_s^2$$

This is the dispersion relation for our rotating thin disk.

The easiest way to demonstrate the desired result is to write the system of three equations in standard form:

$$\begin{aligned} \mathbf{i}k \left( \frac{2\pi G \Sigma_a}{|k|} - \frac{c_s^2}{\Sigma_\theta} \right) \Sigma_a + \mathbf{i}\omega v_{ax} + 2\Omega_\theta v_{ay} &= 0 \\ -\Omega_\theta v_{ax} + \mathbf{i}\omega v_{ay} &= 0 \\ -\mathbf{i}\omega \Sigma_a + \mathbf{i}k \Sigma_\theta v_{ax} &= 0 \end{aligned}$$

We can write this system as a matrix equation:

$$\mathbf{A} \equiv \begin{pmatrix} \mathbf{i}k \left( \frac{2\pi G \Sigma_a}{|k|} - \frac{c_s^2}{\Sigma_\theta} \right) & \mathbf{i}\omega & 2\Omega_\theta \\ \mathbf{0} & -\Omega_\theta & \mathbf{i}\omega \\ -\mathbf{i}\omega & \mathbf{i}k \Sigma_\theta & \mathbf{0} \end{pmatrix}$$

$$\mathbf{A} \begin{pmatrix} \Sigma_a \\ v_{ax} \\ v_{ay} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

This matrix equation has a non-trivial solution if and only if A is non-invertible, i.e., it has zero determinant. Thus the condition for there to be non-trivial solutions we require

$$\begin{aligned} 0 &= \det(\mathbf{A}) \\ &= \mathbf{i}\omega k^2 \Sigma_\theta \left( \frac{2\pi G \Sigma_a}{|k|} - \frac{c_s^2}{\Sigma_\theta} \right) + \mathbf{i}\omega^3 - 2\mathbf{i}\omega \Omega_\theta^2 \\ &= k^2 \Sigma_\theta \left( \frac{2\pi G \Sigma_a}{|k|} - \frac{c_s^2}{\Sigma_\theta} \right) + \omega^3 - 2\Omega_\theta^2 \\ \omega^2 &= 2\Omega_\theta^2 - 2\pi G \Sigma_\theta |k| + k^2 c_s^2 \end{aligned}$$

This is the desired dispersion relation.

e)

Solutions with  $\omega^2 > 0$  correspond to oscillations, while those with  $\omega^2 < 0$  correspond to pairs of modes, one of which decays with time and one of which grows. We refer to the growing modes as unstable, since in the linear regime they become arbitrarily large. Show that an unstable mode exists if  $Q < 1$ , where

$$Q = \frac{\sqrt{2} \Omega_\theta c_s}{\pi G \Sigma_\theta}$$

is called the Toomre parameter. Note that this stability condition refers only to axisymmetric modes in infinitely thin disks; non-axisymmetric instabilities in finite thickness disks usually appear around  $Q \approx 1.5$ .

Instability requires that  $\omega^2 < 0$ , which requires

$$0 > 2\Omega_\theta^2 - 2\pi G \Sigma_\theta |k| + k^2 c_s^2$$

We therefore want to find the value of  $k$  that produces the minimum value of the right-hand side. The RHS is quadratic in  $|k|$ , and its minimum occurs at

$$|k| = \frac{\pi G \Sigma_\theta}{c_s^2}$$

$$In[*]:= \text{Minimize}[2 \Omega_0^2 - 2 \pi G \Sigma_0 k + k^2 c_s^2, k]$$

$$Out[*]:= \left\{ \begin{array}{l} -\infty \\ 2 \Omega_0^2 \\ \frac{-G^2 \pi^2 \Sigma_0^2 + 2 c_s^2 \Omega_0^2}{c_s^2} \end{array} \right. \text{ True} \left\{ \begin{array}{l} (\Sigma_0 > 0 \ \&\& \ c_s == 0 \ \&\& \ G > 0) \ || \ (\Sigma_0 > 0 \ \&\& \ c_s == 0 \ \&\& \ G < 0) \ || \\ (\Sigma_0 < 0 \ \&\& \ c_s == 0 \ \&\& \ G > 0) \ || \ (\Sigma_0 < 0 \ \&\& \ c_s == 0 \ \&\& \ G < 0) \\ (\Sigma_0 == 0 \ \&\& \ c_s == 0) \ || \ (\Sigma_0 == 0 \ \&\& \ c_s > 0) \ || \ (\Sigma_0 == 0 \ \&\& \ c_s < 0) \ || \\ (\Sigma_0 > 0 \ \&\& \ c_s == 0 \ \&\& \ G == 0) \ || \ (\Sigma_0 < 0 \ \&\& \ c_s == 0 \ \&\& \ G == 0) \end{array} \right. , \{k \rightarrow \left\{ \begin{array}{l} \frac{G \pi \Sigma_0}{c_s^2} \\ 0 \\ \text{Indetern} \end{array} \right.$$

Plugging this in, we see that the minimum value of the RHS is given by

$$2 \Omega_0^2 - 2 \pi G \Sigma_0 \frac{\pi G \Sigma_0}{c_s^2} + \left( \frac{\pi G \Sigma_0}{c_s^2} \right)^2 c_s^2$$

Instability exists only if there is a value of  $|k|$  that makes the RHS negative, so the condition is

$$2 \Omega_0^2 - 2 \pi G \Sigma_0 \frac{\pi G \Sigma_0}{c_s^2} + \left( \frac{\pi G \Sigma_0}{c_s^2} \right)^2 c_s^2 < 0$$

$$2 \Omega_0^2 < \frac{\pi G \Sigma_0}{c_s^2}$$

$$\left( \frac{\sqrt{2} \Omega_0 c_s}{\pi G \Sigma_0} \right)^2 < 1$$

$$Q < 1$$

f)

When an unstable mode exists, we define the Toomre wave number  $k_T$  as the wave number that corresponds to mode for which the instability grows fastest. Calculate  $k_T$  and the corresponding Toomre wavelength,  $\lambda_T = 2 \pi / k_T$ .

g)

The Toomre mass, defined as  $M_T = \lambda_T^2 \Sigma_0$ , is the characteristic mass of an unstable fragment produced by Toomre instability. Compute  $M_T$ , and evaluate it for  $Q = 1$ ,  $\Sigma_0 = 12 M_\odot \text{ pc}^{-2}$  and  $c_s = 6 \text{ km s}^{-1}$ , typical values for the atomic ISM in the solar neighborhood. Compare the mass you find to the maximum molecular cloud mass observed in the

The Toomre mass is

$$M_T = \lambda_T^2 \Sigma_0 = \left( \frac{2 \pi}{k_T} \right)^2 \Sigma_0 = \left( \frac{2 \pi}{\frac{\pi G \Sigma_0}{c_s^2}} \right)^2 \Sigma_0 = \frac{4 c_s^2}{G^2 \Sigma_0}$$

Plugging in the given values of  $c_s$  and  $\Sigma_0$ , we obtain  $M_T = 2.3 \times 10^7 M_\odot$ . This is a bit larger than the truncation masses reported by Rosolowsky, but only by a factor of a few.