



Stern- und
Planetenentstehung
Sommersemester 2020
Markus Röllig

Lecture 3: Gas Flows and Turbulence



http://exp-astro.physik.uni-frankfurt.de/star_formation/index.php

VORLESUNG/LECTURE

Raum: Physik - 02.201a

dienstags, 12:00 - 14:00 Uhr

SPRECHSTUNDE:

Raum: GSC, 1/34, Tel.: 47433, (roellig@ph1.uni-koeln.de)

dienstags: 14:00-16:00 Uhr

Nr.	Thema	Termin
1	Observing the cold ISM	21.04.2020
2	Observing Young Stars	28.04.2020
3	Gas Flows and Turbulence Magnetic Fields and Magnetized Turbulence	05.05.2020
4	Gravitational Instability and Collapse	12.05.2020
5	Stellar Feedback	19.05.2020
6	Giant Molecular Clouds	26.05.2020
7	Star Formation Rate at Galactic Scales	02.06.2020
8	Stellar Clustering	09.06.2020
9	Initial Mass Function – Observations and Theory	16.06.2020
10	Massive Star Formation	23.06.2020
11	Protostellar disks and outflows – observations and theory	30.06.2020
12	Protostar Formation and Evolution	07.07.2020
13	Late Stage stars and disks – planet formation	14.07.2020

3 GAS FLOWS AND TURBULENCE

3.1 CHARACTERISTIC NUMBERS FOR FLUID FLOW

3.1.1 The Conservation Equations

Fluids are governed by a series of conservation laws

conservation of mass:

$$\frac{\partial}{\partial t} \rho = -\nabla \cdot (\rho \vec{v})$$

change of mass density at a fixed point is equal to minus the divergence of density times velocity = rate at which mass flows into or out of an infinitesimal volume around that point

conservation of momentum:

$$\frac{\partial}{\partial t} (\rho \vec{v}) = -\nabla \cdot (\rho \vec{v} \vec{v}) - \nabla P + \rho \nu \nabla^2 \vec{v}$$

($\vec{v} \vec{v}$: tensor product, (outer product) -> index notation)

$$\frac{\partial}{\partial t} (\rho v_i) = - \frac{\partial}{\partial x_j} \cdot (\rho v_i v_j) - \frac{\partial}{\partial x_i} P + \rho \nu \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_j} v_i \right)$$

viscosity ν

$\rho \vec{v}$: density of momentum at a point

$\nabla \cdot (\rho \vec{v} \vec{v})$: analog to cons. of mass rate at which momentum is advected into or out of the point by the flow

∇P : rate at which pressure forces acting on the fluid changes its momentum

$\rho \nu \nabla^2 \vec{v}$: rate at which viscosity redistributes momentum (This is the only term without analogous counterpart in Newton's second law for single particles)

ν : kinematic viscosity

The viscous term describes the change in fluid momentum due to diffusion of momentum from adjacent fluid elements.

fluid velocity = systematic flow + random velocity field ↗

viscosity turns bulk motion into random motion => dissipation

3.1.2 The Reynolds Number and the Mach Number

To understand rel. importance of terms in momentum equation

⇒ dimensional analysis

system of characteristic size L and characteristic velocity V

(mol. cloud: $L \sim 10 \text{ pc}$, $V \sim 5 \text{ km s}^{-1}$)

the natural time scale for such a system: L/V

$$\Rightarrow \left(\frac{\partial X}{\partial t} \sim \frac{X}{\frac{L}{V}} \right) \quad \text{and} \quad \left(\frac{\partial X}{\partial x} \sim \frac{X}{L} \right)$$

momentum equation ($P = \rho c_s^2$, c_s = gas sound speed):

$$\frac{\rho V^2}{L} \sim \frac{\rho V^2}{L} + \frac{\rho c_s^2}{L V^2} + \rho \frac{V}{V L^2}$$
$$1 \sim 1 + \frac{c_s^2}{V^2} + \frac{v}{V L}$$

$/: \frac{V^2}{L}$

$\uparrow \frac{1}{M^2} \quad \uparrow \frac{1}{Re} \sim \nu$

We define:

$$\text{Mach number} \quad \mathcal{M} \sim \frac{V}{c_s}$$

$$\text{Reynolds number} \quad Re \sim \frac{LV}{\nu}$$

If $\mathcal{M} \ll 1$ then $\frac{c_s^2}{V^2} \gg 1$ pressure term is important

$\mathcal{M} \gg 1$ pressure term unimportant

$$c_s = \sqrt{\frac{k_B T}{\mu m_H}} = 0.18 \left(\frac{T}{10K} \right)^{1/2}, \quad \mu = 2.33 \text{ (mean mass per particle)}$$

assuming 1He per 10 H $\Rightarrow 14/6$)

$$\Rightarrow \frac{MV}{c_s} \sim 20 \quad \Rightarrow \text{pressure forces unimportant in mol. clouds}$$

If $Re \lesssim 1$ viscous forces are important

$Re \gg 1$ viscous forces are unimportant

Reynolds number describes characteristic length scale $L \sim \nu/V$ in the flow

This is the length scale on which diffusion causes the flow to dissipate energy \Rightarrow large scale motion effectively dissipationless.

ideal gas: $\nu = 2\bar{u}\lambda$ \bar{u} : RMS molecular speed (order of c_s)

λ : particle mean free path

(inverse of cross-section times density)

$$\lambda \sim \frac{1}{\sigma n} \sim [1(nm)^2 (100cm^{-3})]^{-1} \sim 10^{12} \text{ cm}$$

$$\nu \sim 10^{16} \text{ cm}^2 \text{ s}^{-1} \text{ and } Re \sim 10^9$$

\Rightarrow viscous forces are unimportant in molecular clouds

⇒ molecular clouds must be highly turbulent! (turbulent if $Re \gtrsim 10^3 - 10^4$)

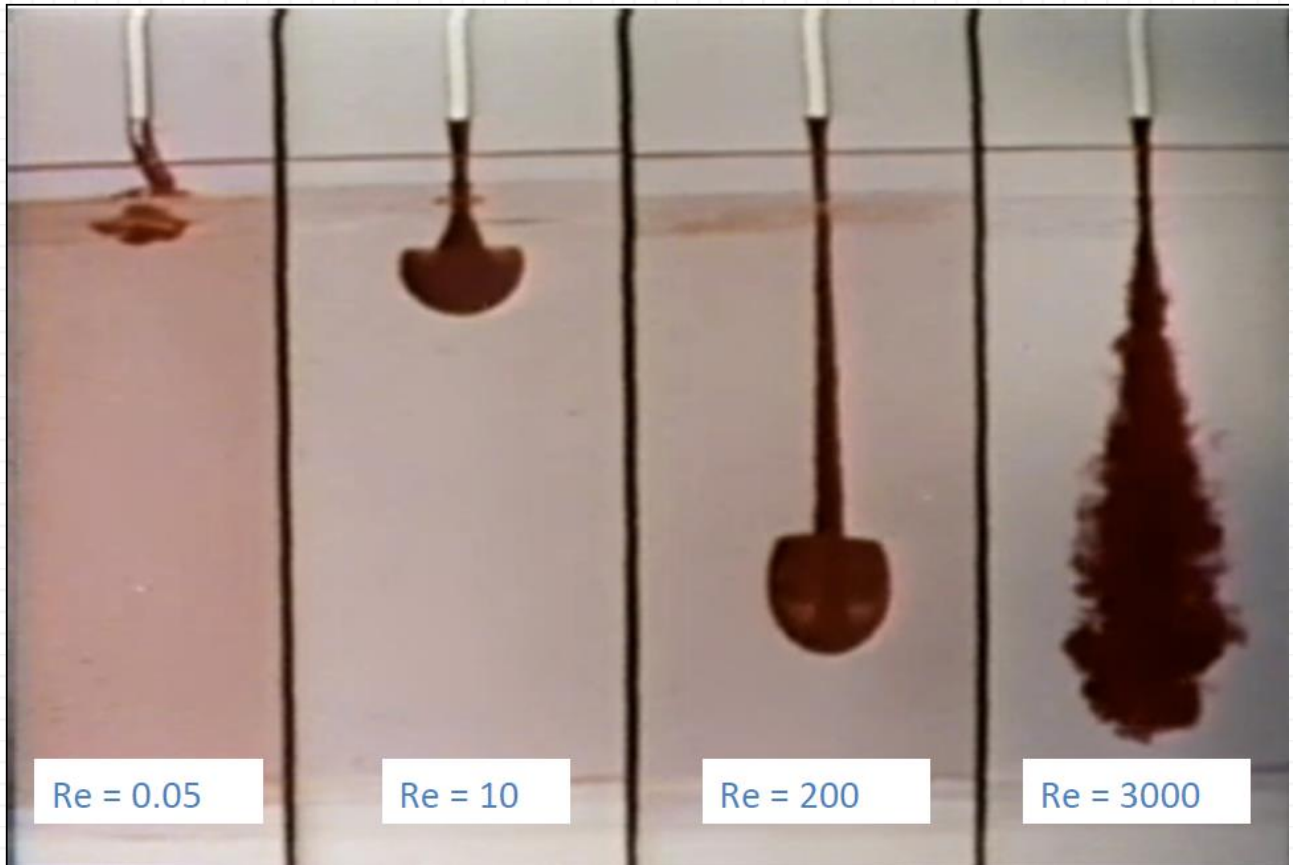


Abbildung 1 National Committee for Fluid Mechanics Films (NCFMF) / Taylor 1964

(see video)

3.2 MODELLING TURBULENCE

3.2.1 Velocity Statistics

Important property of turbulence: velocity structure, i.e. how does the velocity change from point to point

In turbulent medium, velocity fluctuates in time and space => statistical study

Assumptions:

- homogeneous turbulence (turbulent motions vary only randomly, not systematically)
- isotropic turbulence (no preferred direction)
(neither are true for mol. clouds!)

Autocorrelation function characterizes how \vec{v} varies with position

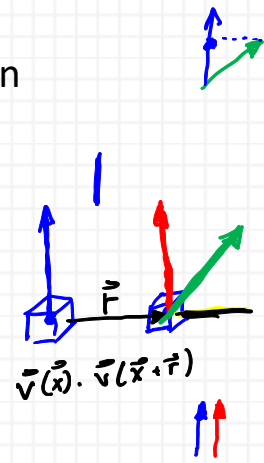
$$A(\vec{r}) = \frac{1}{V} \int \vec{v}(\vec{x}) \cdot \vec{v}(\vec{x} + \vec{r}) dV \equiv \langle \vec{v}(\vec{x}) \cdot \vec{v}(\vec{x} + \vec{r}) \rangle$$

$\langle \dots \rangle$ indicate average over all positions

$A(0) = \langle |\vec{v}|^2 \rangle$ RMS velocity in the fluid

Isotropicity: $A(\vec{r})$ cannot depend on direction $\rightarrow \vec{r} \rightarrow |\vec{r}|$

$A(r)$ measures how different the velocity is at a scale r



Alternatively, in Fourier space: the Fourier transform of the velocity field:

$$\tilde{v}(\vec{k}) = \frac{1}{\sqrt{2\pi}} \int \vec{v}(\vec{x}) e^{i\vec{k} \cdot \vec{x}} d\vec{x}$$

The the power spectrum is:

$$\Psi(\vec{k}) = |\tilde{v}(\vec{k})|^2$$

Again, because of isotropicity, the power spectrum depends only on the magnitude of the wave number $k = |\vec{k}|$, not its direction.

Instead of Ψ we use the power per unit radius in k -space:

$$P(\vec{k}) = 4\pi k^2 \Psi(k)$$

(Total power integrated over some shell from k to dk in k -space.)

Parseval's Theorem

$$\int P(k) dk = \int |\tilde{v}(\vec{k})|^2 d^3\vec{k} = \int \vec{v}(\vec{x})^2 d^3\vec{x}$$

Integral over power spectral density over all wavenumbers is equal to the integral of the square of the velocity over all space

\Rightarrow if $\rho = \text{const}$ the integral of the power spectrum is the kinetic energy per unit mass in the flow!

Wiener-Khinchin Theorem

$P(k)$ is just the Fourier transform of the autocorrelation function

$$\Psi(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int A(\vec{r}) e^{i\vec{k}\cdot\vec{r}} d\vec{r}$$

- ⇒ Power spectrum at wavenumber $k \rightarrow$ fraction of total power is in motions at that wavenumber, i.e. on that characteristic length scale ($k = 2\pi/\lambda$).
- ⇒ A power spectrum that peaks at low $k \rightarrow$ most of the turbulent power is in large-scale motions
- ⇒ A power spectrum that peaks at high $k \rightarrow$ most of the turbulent power is in small-scale motions
- ⇒ The power spectrum also specifies how the velocity dispersions varies when measured over a region of some characteristic size.

Consider a volume of size l and a velocity dispersion $\sigma_v(l)$ within it.

Suppose that the power spectrum is described by a power law

$P(k) \propto k^{-n}$. The total kinetic energy per unit within the region is

$$KE \sim \sigma_v(l)^2$$

Above, we saw that we can also write the kinetic energy in terms of the power spectrum integrating over those modes that are small enough to fit within the volume

$$KE \sim \int_{2\pi/l}^{\infty} P(k) dk \propto l^{n-1}$$

normalizing, by defining the sonic scale l_s as the size of a region within which the velocity dispersion is equal to the thermal sound speed of gas, it follows that:

$$\sigma_v = c_s \left(\frac{l}{l_s} \right)^{\frac{n-1}{2}}$$

3.2.2 The Kolmogorov Model and Turbulence Cascades

Basic theory of subsonic, hydrodynamic turbulence: Kolmogorov (1941)
(translation: Kolmogorov (1991))

- turbulence occurs when Re is large (large range of scales where dissipation is unimportant)
- for incompressible motion transfer of energy can only occur between adjacent wavenumbers
 - energy at a scale k cannot be transferred directly to some scale $k' \ll k$
 - instead it must cascade through intermediate scales between k and k'

Energy is injected into a system on some large scale that is dissipationless and it cascades down to smaller scales until at some scale $Re \sim 1$ at which dissipation becomes important.

If turbulence is in equilibrium, the energy at some scale k depends only on k and on the rate of injection (or dissipation, which are equal) ψ .

Dimensional analysis:

$k \sim 1/L$, power spectrum \sim energy per unit mass ($\sim v^2 = L^2/T^2$) per unit radius in k -space ($\sim 1/L$) $P(k) \sim L^3/T^2$, $\psi \sim L^2/T^3$

$P(k)$ is a function of k and ψ we write (for some dimensionless C)

$$P(k) = C k^\alpha \psi^\beta$$

Then

$$\frac{L^3}{T^2} \sim L^{-\alpha} \left(\frac{L^2}{T^3} \right)^\beta$$

$$L^3 \sim L^{-\alpha+2\beta}$$

$$T^{-2} \sim T^{-3\beta}$$

$$\beta = \frac{2}{3}$$

$$\alpha = 2\beta - 3 = -\frac{5}{3}$$

- power in the flow varies with energy injection rate to the 2/3
- power distributed such that the power at a given wavenumber k varies as $k^{-5/3}$
 - most power in the largest scale motion
- we find that $\sigma_v \propto l^{1/3}$
 - velocity dispersion increases with size scale as size to the 1/3
 - linewidth-size relation
 - larger regions should have larger linewidths

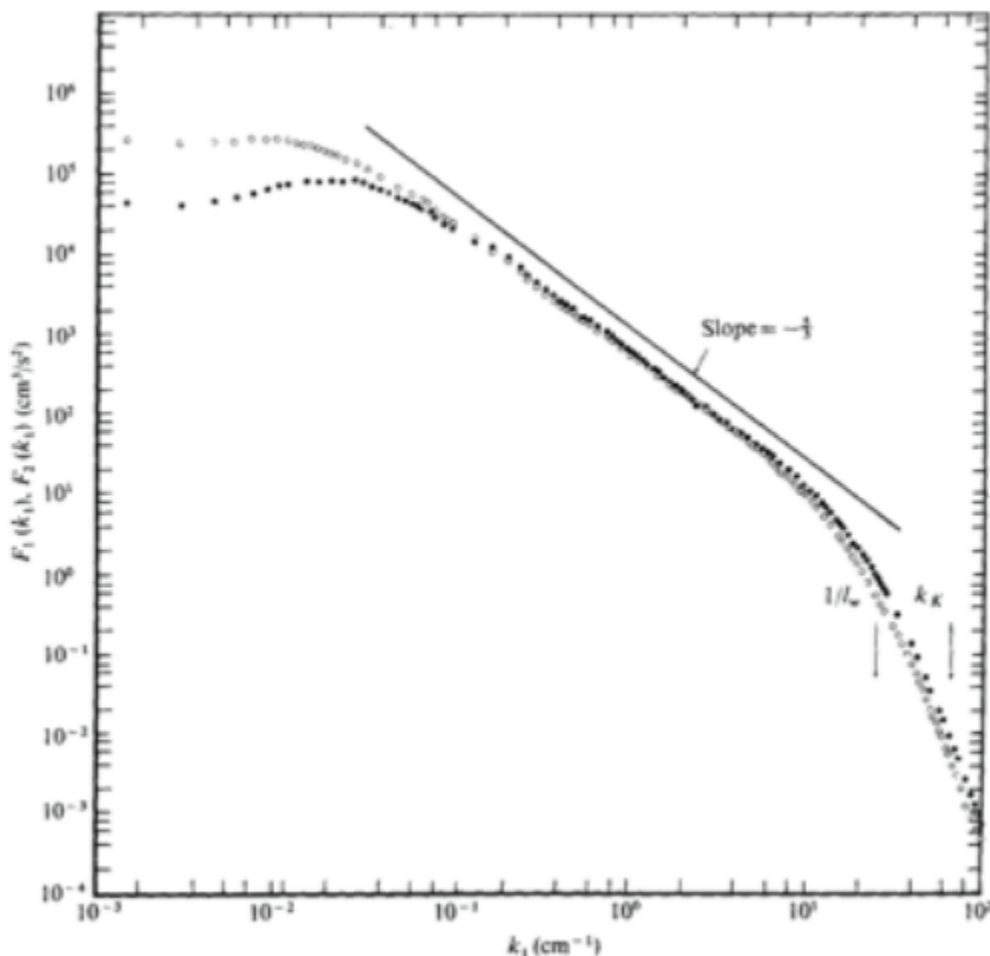


Abbildung 2 Power spectrum for turbulence generated in an air jet (Champagne 1978)

3.3 SUPERSONIC TURBULENCE

3.3.1 Velocity Statistics

real interstellar clouds: $Re \gg 1$, $\mathcal{M} \gg 1$ supersonic
pressure unimportant on scales $L \gg l_s$
viscosity also unimportant on large scales

- gas tends to move ballistically on large scales
- on small scales sharp velocity gradients (fast volumes will overtake slow volumes)
- eventually viscosity will stop the fluid from moving ballistically
 - ⇒ shocks (size scale determined by viscosity)

velocity field: series of step functions
power spectrum: $P(k) \propto k^{-2}$

subsonic turbulence: $-5/3$ energy decay via cascade to small scales

supersonic turbulence -2 decay via shock formation

(one single shock generates $P(k) \propto k^{-2}$. i.e. nonlocally couples many scales

- ⇒ all scales are coupled in shocks

3.3.2 Density Statistics

in subsonic flows, the dominant pressure leads to ~constant density in isothermal gas

in supersonic flows -> highly compressible -> structure of density field

From numerical experiments and empirical arguments:

supersonically turbulent, isothermal medium -> log-normal distribution

$$p(s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(s-s_0)^2}{2\sigma_s^2}\right], \quad \text{where } s = \ln\left(\frac{\rho}{\bar{\rho}}\right)$$

s : log of density normalized to mean density $\bar{\rho}$, s_0 : median of distribution

probability that the density at a randomly chosen point will be such that $\ln(\rho/\bar{\rho})$ is in the range from s to $s + ds$.

Because $\bar{\rho} = \int p(s)\rho ds = e^{s_0+\sigma_s^2/2}$ we find: $s_0 = -\sigma_s^2/2$

probability that a randomly chosen mass element will have a particular density:

consider a volume V , all material with density such that $\ln(\rho/\bar{\rho})$ is in the range from s to $s + ds$. This material occupies a volume $dV = p(s)V$ and has a mass:

$$\begin{aligned} dM &= \rho p(s)dV \\ &= \bar{\rho} e^s \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(s-s_0)^2}{2\sigma_s^2}\right] dV \\ p_M(s) &= \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(s+s_0)^2}{2\sigma_s^2}\right] dV \end{aligned}$$

The mass PDF is the same as the volume PDF but with the peak shifted from $-s_0$ to $+s_0$:

- ⇒ typical volume element is at a density lower than the mean (because mass is collected into shocks)
- ⇒ typical mass element is in those shocked regions and is at a higher than average density

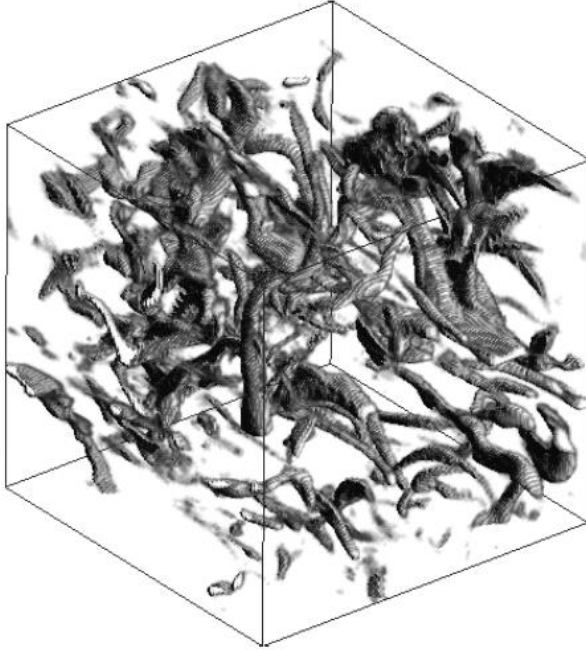


Abbildung 3 Density field of supersonic turbulence (Padoan Z& Nordlund 1999)

Supersonic turbulence:

series of shocks and rarefaction

density multiplied by factors > and < 1

A log-normal process is the statistical realization of the multiplicative product of many independent random variables, each of which is positive.

log-normal in density

dispersion of densities (empirically from num. simulations)

$$\sigma_s^2 \approx \ln \left(1 + b^2 \mathcal{M}^2 \frac{\beta_0}{\beta_0 + 1} \right)$$

b is a number in the 1/3-1 (depends on how compressive vs. selenoidal) the velocity field is

β_0 is the ratio of thermal to magnetic pressure at the mean density and magnetic field strength

4 MAGNETIC FIELDS AND MAGNETIZED TURBULENCE
