Submillimeter Observations of dust in the continuum

D = 450 pc
$\varnothing_{beam}$ = 10''
$S(\lambda = 350\ \mu m) = 1060$ Jy
$S(\lambda = 450\ \mu m) = 410$ Jy

For the flux S_λ at a given wavelength λ we find

$$S_\lambda = B_\lambda(T_{dust})\,(1 - \exp(-\tau_\lambda))\,\Omega_{beam} \quad (1)$$

with the optical depth

$$\tau_\lambda = M_d / M_g\, \sigma_\lambda\, N_H \quad \text{with } M_d / M_g = 0.01 \quad (2)$$

and $\sigma_\lambda = 7 \times 10^{-21}\, \lambda^{-\beta}$ (with λ in μm!) as dust absorption cross section in units of cm^2/H – atom and N_H is the Hydrogen column density in cm^{-2}. The spectral dust index β describes the variation of the dust emission with λ, Ω_{beam} is the solid angle (in steradian). $B_\lambda(T_{dust})$ is the Planck law for the dust temperature T_{dust}.

Determine T_{dust}

Assuming that the dust is optically thin, determine T_{dust} from the ratio of the measured flux values for β=2.2

1 Jy = $10^{-26}\, W m^{-2}\, Hz^{-1}\, sr^{-1}$

$$B_\nu = \frac{2 h \nu^3}{c^2} \frac{1}{\mathrm{Exp}\left[\frac{h\nu}{kT}\right] - 1} \quad \left(W\, m^{-2}\, Hz^{-1}\, sr^{-1} = Jy\right) \quad (3)$$

$$B_\lambda = \frac{2 h c^2}{\lambda^5} \frac{1}{\mathrm{Exp}\left[\frac{hc}{\lambda k T}\right] - 1} \quad \left(W\, m^{-2}\, m^{-1}\, sr^{-1}\right) \quad (4)$$

$B_\nu \neq B_\lambda$ but $B_\nu\, d\nu = B_\lambda\, d\lambda$

from $c = \lambda\, \nu$ follows $d\lambda = -\frac{c}{\nu^2} d\nu$ and $d\nu = -\frac{c}{\lambda^2} d\lambda$ (ν and λ increase in opposite directions!)

$$B_\nu\, d\nu = \frac{2 h \nu^3}{c^2} \frac{d\nu}{\mathrm{Exp}\left[\frac{h\nu}{kT}\right]-1} = \frac{2 h c^3}{c^2 \lambda^3} \frac{c}{\lambda^2} \frac{d\lambda}{\mathrm{Exp}\left[\frac{hc}{\lambda k T}\right]-1} = \frac{2 h c^2}{\lambda^5} \frac{d\lambda}{\mathrm{Exp}\left[\frac{hc}{\lambda k T}\right]-1} = B_\lambda\, d\lambda$$

and

$$B_\nu = B_\lambda \frac{\lambda^2}{c}$$

optical thin →

$$\tau_\nu = M_d / M_g\, \sigma_\lambda\, N_H = 0.01 \times 7 \times 10^{-21}\, \lambda^{-\beta}\, N_H \quad (5)$$

$$S_\lambda = B_\lambda(T_{dust})\,(1 - \exp(-\tau_\lambda))\,\Omega_{beam} \approx B_\lambda(T_{dust})\,\tau_\lambda\,\Omega_{beam} \quad (6)$$

In[ ]:=
```
Series[(1 - Exp[-τλ]), τλ → 0]
```

Out[ ]= $\tau_\lambda + O[\tau_\lambda]^2$

Then ($S_1 = 1060$ Jy, $S_2 = 410$ Jy)

$$\frac{S_1}{S_2} = \frac{B_{\nu_1}(T_{dust})\, \tau_{\lambda_1}\, \Omega_{beam}}{B_{\nu_2}(T_{dust})\, \tau_{\lambda_2}\, \Omega_{beam}} = \frac{B_{\nu_1}(T_{dust})}{B_{\nu_2}(T_{dust})} \left(\frac{\lambda_1}{\lambda_2}\right)^{-\beta} \tag{7}$$

and

$$\frac{S_1}{S_2} \left(\frac{\lambda_1}{\lambda_2}\right)^{-2} = \frac{B_{\lambda_1}(T_{dust})}{B_{\lambda_2}(T_{dust})} \left(\frac{\lambda_1}{\lambda_2}\right)^{-\beta} \tag{8}$$

Example with Eq. (7)

1. Using B_ν

$$\frac{S_1}{S_2} = \frac{B_{\nu_1}(T_{dust})}{B_{\nu_2}(T_{dust})} \left(\frac{\lambda_1}{\lambda_2}\right)^{-\beta} = \left(\frac{\nu_1}{\nu_2}\right)^3 \frac{\text{Exp}\left[\frac{h\nu_2}{kT}\right] - 1}{\text{Exp}\left[\frac{h\nu_1}{kT}\right] - 1} \left(\frac{\lambda_1}{\lambda_2}\right)^{-\beta}$$

$\nu_1 = c/\lambda_1 = 2.998 \times 10^{10}\ \text{cm s}^{-1} / 350 \times 10^{-4}\ \text{cm} = 8.56571 \times 10^{11}$ Hz

$\nu_2 = c/\lambda_2 = 2.998 \times 10^{10}\ \text{cm s}^{-1} / 450 \times 10^{-4}\ \text{cm} = 6.66222 \times 10^{11}$ Hz

$h = 6.626 \times 10^{-27}$ erg s, $k = 1.381 \times 10^{-16}$ erg K^{-1}

$$\frac{1060}{410} = \left(\frac{8.566}{6.662}\right)^3 \frac{\text{Exp}\left[\frac{6.626 \times 10^{-27}\ 6.66222 \times 10^{11}}{1.381 \times 10^{-16}\ T}\right] - 1}{\text{Exp}\left[\frac{6.626 \times 10^{-27}\ 8.56571 \times 10^{11}}{1.381 \times 10^{-16}\ T}\right] - 1} \left(\frac{350}{450}\right)^{-\beta}$$

$$\frac{S_1}{S_2} = \frac{1060}{410} = \left(\frac{8.566}{6.662}\right)^3 \frac{\text{Exp}\left[\frac{31.9652}{T}\right] - 1}{\text{Exp}\left[\frac{41.0981}{T}\right] - 1} \left(\frac{350}{450}\right)^{-\beta}$$

In[195]:=
```
sol = T /. FindRoot[1060/410 - (8.566/6.662)^3 (Exp[31.965158902566586`/T] - 1)/(Exp[41.09806144615705`/T] - 1) (350/450)^-2.2, {T, 40}]
```

Out[195]= 48.5032

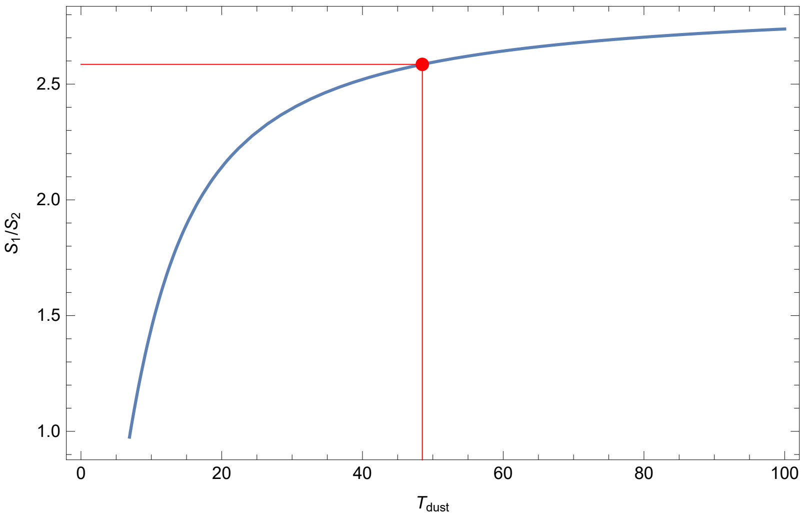
In[ ]:= $$\text{Plot}\left[\left(\frac{8.566}{6.662}\right)^{3}\frac{\text{Exp}\left[\frac{31.965158902566586`}{T}\right]-1}{\text{Exp}\left[\frac{41.09806144615705`}{T}\right]-1}\left(\frac{350}{450}\right)^{-2.2},\{T,0,100\},\right.$$

Frame → True, Axes → False, FrameLabel → {"T_{dust}", "S_1/S_2"}, Epilog → {

$$\left\{\text{PointSize}\to\text{Large},\ \text{Red},\ \text{Point}\left[\left\{\text{sol},\frac{1060}{410}\right\}\right],\right.$$

$$\left.\text{Line}\left[\left\{\left\{0,\frac{1060}{410}\right\},\left\{\text{sol},\frac{1060}{410}\right\}\right\}\right],\ \text{Line}\left[\left\{\left\{\text{sol},\frac{1060}{410}\right\},\{\text{sol},0\}\right\}\right]\right\}\Big]$$

Out[ ]=



In[ ]:= $$\Omega=\pi\left(\left(5/3600.\right)*\frac{\pi}{180}\right)^{2}$$

Out[ ]= 1.84603×10^{-9}

In[196]:= $\nu1 = 8.566\times10^{11}$;

$\nu2 = 6.662\times10^{11}$;

In[198]:=

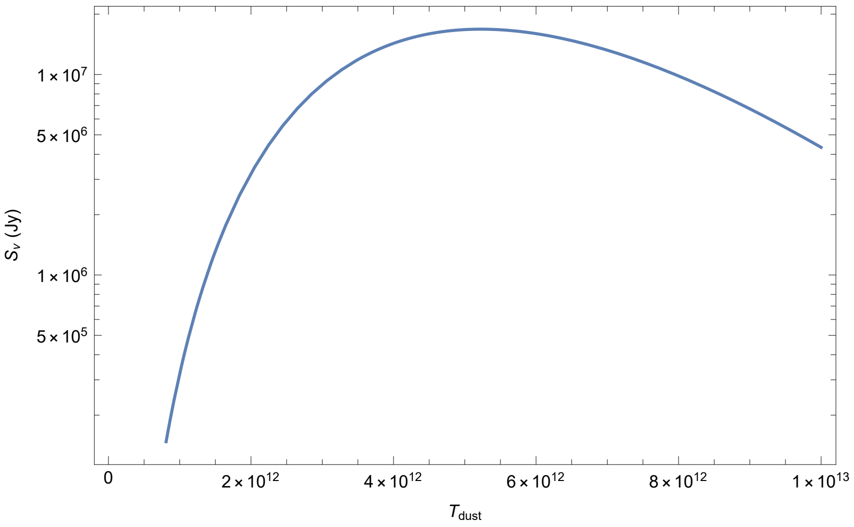
$$\text{planck}\nu[\nu\_, T\_] := \frac{2 * 6.626 \times 10^{-27}\, \nu^3}{(2.998 \times 10^{10})^2} \frac{1}{\text{Exp}\left[\frac{6.626 \times 10^{-27}\, \nu}{1.38 \times 10^{-16}\, T}\right] - 1}$$

$$\text{flux}[\nu\_, T\_, \text{column}\_, \text{solidangle}\_] := \text{planck}\nu[\nu, T]\, 0.01 \times 7 \times 10^{-21} \left(\frac{2.998 \times 10^{10}}{\nu}\right)^{-2.2} \text{column} * \text{solidangle}$$

$$\text{Plot}\left[\text{flux}\left[\nu, \text{sol}, 10^{21}, \left((10/3600.) * \frac{\pi}{180}\right)^2\right] 10^{23}, \{\nu, 10^{10}, 10^{13}\},$$

```
 ScalingFunctions → "Log10", Frame → True, Axes → False, FrameLabel → {"Tdust", "Sν (Jy)"}]
```

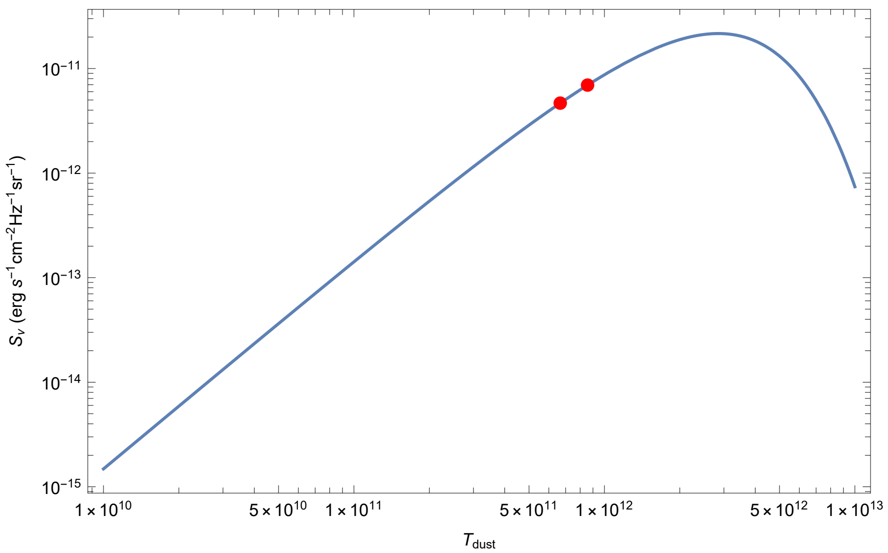
Out[200]=



In[∘]:=

```
Plot[planckν[ν, sol], {ν, 10^10, 10^13}, Frame → True,
 Axes → False, FrameLabel → {"Tdust", "Sν (erg s^-1cm^-2Hz^-1sr^-1)"},
 ScalingFunctions → {"Log", "Log"},
 Epilog → {Red, PointSize → Large,
   Point[Log@{ν1, planckν[ν1, sol]}], Point[Log@{ν2, planckν[ν2, sol]}]}]
```

Out[∘]=



2. Using B_λ

$$\frac{S_1}{S_2}\left(\frac{350}{450}\right)^{-2} = \left(\frac{\lambda_1}{\lambda_2}\right)^{-5} \frac{\text{Exp}\left[\frac{h\,c}{\lambda_2\, k\ T}\right]-1}{\text{Exp}\left[\frac{h\,c}{\lambda_1\, k\ T}\right]-1}\left(\frac{\lambda_1}{\lambda_2}\right)^{-\beta} = \frac{\text{Exp}\left[\frac{h\,c}{\lambda_2\, k\ T}\right]-1}{\text{Exp}\left[\frac{h\,c}{\lambda_1\, k\ T}\right]-1}\left(\frac{\lambda_1}{\lambda_2}\right)^{-\beta-5}$$

In[201]:=
```
h = 6.626 × 10^-27;
k = 1.38 × 10^-16;
c = 2.998 × 10^10;
λ1 = 350 × 10^-4;
λ2 = 450 × 10^-4;
β = 2.2;
```

In[ ]:=

$$\text{FindRoot}\left[\frac{1060}{410}\left(\frac{350}{450}\right)^{-2} - \frac{\text{Exp}\left[\frac{h\,c}{\lambda_2\, k\ T}\right]-1}{\text{Exp}\left[\frac{h\,c}{\lambda_1\, k\ T}\right]-1}\left(\frac{\lambda_1}{\lambda_2}\right)^{-\beta-5},\ \{T, 40\}\right]$$

Out[ ]= {T → 48.6213}

Determine column density N_H and M_{dust}

$S = B_\lambda\, \tau_\lambda\, \Omega = B_\lambda\, 0.01\, \sigma_\lambda\, N_H\, \Omega$

$$N_H = \frac{S\,c / \lambda^2}{B_\lambda\, 0.01\, \sigma_\lambda\, \Omega} = \frac{S}{B_\nu\, 0.01\, \sigma_\lambda\, \Omega} \qquad (9)$$

$S = 1060\ \text{Jy} = 1060 \times 10^{-26}\ \text{W} / \text{m}^2 / \text{Hz} = 1060 \times 10^{-23}\ \text{erg} / \text{s} / \text{cm}^2 / \text{Hz}$

In[ ]:=

$$\frac{1060 \times 10^{-23}}{\text{planck}\nu[\nu 1,\ \text{sol}]\ 0.01 \times 7 \times 10^{-21} \times 350^{-2.2}\ \Omega}$$

Out[ ]= 4.67104×10^{27}

Unit check

$$\frac{\text{erg s}^{-1}\ \text{cm}^{-2}\ \text{Hz}^{-1}}{\text{erg s}^{-1}\ \text{cm}^{-2}\ \text{Hz}^{-1}\ \text{sr}^{-1}\ \text{cm}^2\ \text{sr}} = \text{cm}^{-2}\ \text{OK!}$$

The number is actually much too high!

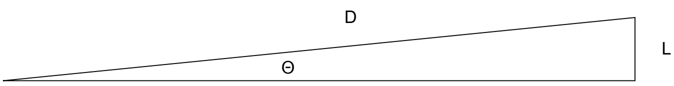
Now we compute the mass from the column density

$$M = N_H\, m_H\, \Theta_x\, \Theta_y\, D^2 \qquad (10)$$

To understand why, look at the geometry:

In[ ]:=
```
Graphics[{Line[{{0, 0}, {10, 0}, {10, 1}, {0, 0}}],
  Text["L", {10.5, 0.5}], Text["D", {5.5, 1}], Text["Θ", {4.5, 0.2}]}]
```

Out[ ]=



then

$$L = \Theta \times D,\quad \Omega = \pi\ (5\ ")^2 \text{sr}{=}1.8460336577110374\text{`}*\text{^}{-9}\ \text{sr} \qquad (11)$$

In[ ]:=

$$\frac{5.}{3600}\ \text{Degree} * 450 \times 3.08 \times 10^{18}$$

Out[ ]= 3.35976×10^{16}

L is about 0.01 pc

$m_H = 1.67 \times 10^{-24}\, g,\ \ D = 450\ \text{pc} = 450 * 3.08 \times 10^{18}\ \text{cm} = 1.386 \times 10^{21}$

$$M = N_H\, m_H\, \Theta_x\, \Theta_y\, D^2 = 1.67 \times 10^{-24} * 4.67104 \times 10^{27} * \Omega * \left(1.386 \times 10^{21}\right)^2 = 2.76628 \times 10^{37}$$

In[ ]:=
$$\frac{2.77 \times 10^{37}}{2 \times 10^{33}}$$

Out[ ]= 13 850.

Too high...

Molecular Line Emission Analysis

From LTE condition follows

$$\kappa_\nu = \frac{c^2}{8\pi} \frac{1}{\nu^2} \frac{g_2}{g_1} n_1 A_{21} \left(1 - \mathrm{Exp}\left[-\frac{h\nu}{k\, T_{ex}}\right]\right) \quad (12)$$

in LTE, there is one excitation temperature T_{ex} that describes the level population of the molecule in the Boltzmann distribution

Changing from level population to column density $n_1 \to N_1$ by integration along the line of sight

$$N_1 = \int n_1\, ds$$

which corresponds to a change from absorption coefficient to optical depth, because

$$\tau_\nu = \int \kappa_\nu\, ds$$

therefore

$$\tau_\nu = \frac{c^2}{8\pi} \frac{1}{\nu^2} \frac{g_2}{g_1} N_1 A_{21} \left(1 - \mathrm{Exp}\left[-\frac{h\nu}{k\, T_{ex}}\right]\right) \quad (13)$$

We now change the variable from frequency to velocity v, integrate over the entire line profile and solve for the column density in one (e.g. lower) level

(with $dv = \frac{c}{freq}\, d(freq)$)

$$N_1 = \frac{8\pi\nu^3}{c^3} \frac{g_1}{g_2} \frac{1}{A_{21}\left(1 - \mathrm{Exp}\left[-\frac{h\nu}{k\, T_{ex}}\right]\right)} \int \tau\, dv = 93.5 \frac{\nu^3\,[\mathrm{GHz}]}{A_{21}} \frac{g_1}{g_2} \frac{1}{1 - \mathrm{Exp}\left[-\frac{0.048\,\nu}{T_{ex}}\right]} \int \tau\, dv \quad (14)$$

Assumption: LTE, T_{ex} known , optically thin

Simplifications: observed integrated line intensity related to T_{ex} and τ by

$$T_{MB}\, \Delta v \approx T_{ex}\, \tau\, \Delta v \approx T_{ex} \int \tau\, dv$$

where Δv = FWHM linewidth. The column density then becomes

$$N_i \approx 1.94 \times 10^3 \frac{v^2}{A_{ji}} T_{MB}\, \Delta v \quad (15)$$

and

$$N_i \approx \frac{g_i\, N_{tot}}{Z} \mathrm{Exp}\left[-E_i / T\right] \quad (16)$$

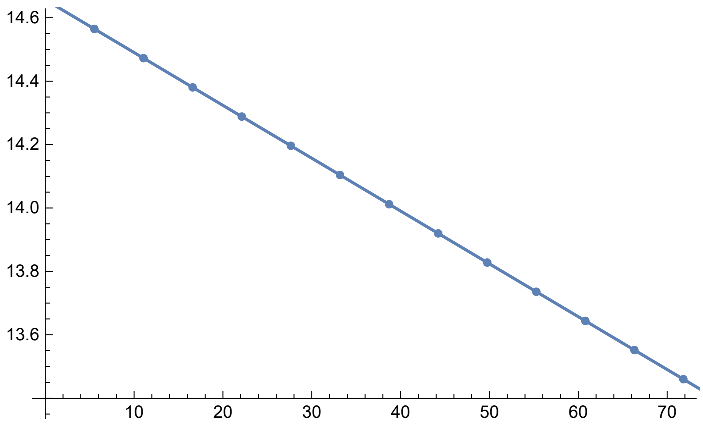
gives

$$\frac{N_{tot}}{Z} \text{Exp}[-E_i / T] = K\, T_{MB}\, \Delta v \tag{17}$$

$K = 1.94 \times 10^3 \frac{v^2}{g_i A_{ji}}$. Taking the log gives

$$\text{Log}[N_{tot}] - \text{Log}[Z] - \text{Log}[K] - \frac{E_i}{T} = \text{Log}[T_{MB}\, \Delta v] \tag{18}$$

```
In[21]:= energiesK = (6.626 × 10^-27)/(1.381 × 10^-16) frequencyList["CO"][[1 ;; 13]] 10^9

Out[21]= {5.53068, 11.0611, 16.5912, 22.1206, 27.6492, 33.1767,
  38.7029, 44.2277, 49.7508, 55.2719, 60.791, 66.3078, 71.8219}

In[116]:= LTEdata = Log10[10^16] - Log10[partitionFunction["CO", 60]] -
   (*Log[1.94 10^3 frequencyList["CO"][[1;;13]]^2/(statWeights EinsteinAList["CO"][[1;;13]])]-*) energiesK/60

Out[116]= {14.5648, 14.4726, 14.3805, 14.2883, 14.1962, 14.104,
  14.0119, 13.9199, 13.8278, 13.7358, 13.6438, 13.5519, 13.46}

In[120]:= FindFit[Transpose[{energiesK, LTEdata}], m x + b, {{m, 0.1}, b}, x]

Out[120]= {m → -0.0166667, b → 14.657}

In[121]:= Show[{
   ListPlot[Transpose[{energiesK, LTEdata}]],
   Plot[-0.01666666666666661` e + 14.65699313955843`, {e, 0, 100}]}]

Out[121]=
```

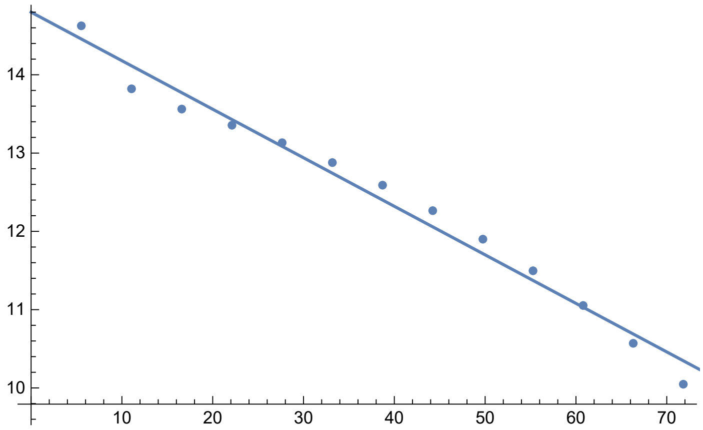


Now with "better" LTE intensity data

```
In[113]:= FindFit[
  Transpose[{energiesK, Log10[1.94 × 10^3 frequencyList["CO"][[1 ;; 13]]^2/(statWeights EinsteinAList["CO"][[1 ;; 13]])
      (LTEBrightnessTemperature[{"CO", #}, {60, 10^15, 0}] & /@ Range[13])]}], m
   x + b, {{m, 20.}, b}, x]

Out[113]= {m → -0.0620688, b → 14.8091}
```

In[101]:=
```
Show[{ListPlot[
   Transpose[{energiesK, Log10[1.94 × 10^3 frequencyList["CO"][[1 ;; 13]]^2 / (statWeights EinsteinAList["CO"][[1 ;; 13]])
        (LTEBrightnessTemperature[{"CO", #}, {60, 10^15, 0}] & /@ Range[13])]}]],
  Plot[-0.062 e + 14.8, {e, 0, 100}]}]
```

Out[101]=



Radex example

From RADEX (T=60K, n=1e4, N=1e16, Δv=1 km/s)

In[7]:=
```
radex = {{1, "--", 0, 115.2712`, 618.629`, 0.01013`, 6.199`},
    {2, "--", 1, 230.538`, 39.92`, 0.5457`, 14.49`}, {3, "--", 2, 345.796`,
     27.573`, 1.158`, 13.76`}, {4, "--", 3, 461.0408`, 22.103`, 1.181`, 8.905`},
    {5, "--", 4, 576.2679`, 19.422`, 0.6499`, 4.191`}, {6, "--", 5, 691.4731`,
     20.153`, 0.1992`, 1.431`}, {7, "--", 6, 806.6518`, 22.726`, 0.04526`, 0.3813`},
    {8, "--", 7, 921.7997`, 25.631`, 0.009435`, 0.08996`},
    {9, "--", 8, 1036.9124`, 27.912`, 0.001905`, 0.01915`},
    {10, "--", 9, 1151.9855`, 29.74`, 0.0003596`, 0.00367`},
    {11, "--", 10, 1267.0145`, 31.619`, 0.00006211`, 0.0006464`},
    {12, "--", 11, 1381.9951`, 33.454`, 9.949`*^-6, 0.0001054`},
    {13, "--", 12, 1496.9229`, 35.758`, 1.483`*^-6, 0.0000165`}};
data = radex[[All, -1]]
```

Out[8]=
```
{6.199, 14.49, 13.76, 8.905, 4.191, 1.431, 0.3813,
 0.08996, 0.01915, 0.00367, 0.0006464, 0.0001054, 0.0000165}
```

In[1]:=
```
Needs["LTE`"]
```

In[3]:=
```
frequencyList[spec_] := Normal[(QuantityMagnitude /@
      Normal[lamdaData[spec, "Transitions", All, "Frequency"]])][[All, 2]]
EinsteinAList[spec_] := Normal[(QuantityMagnitude /@
      Normal[lamdaData[spec, "Transitions", All, "EinsteinA"]])][[All, 2]]
```

In[5]:=
```
frequencyList["CO"][[1 ;; 13]]
EinsteinAList["CO"][[1 ;; 13]]
```

Out[5]=
```
{115.271, 230.538, 345.796, 461.041, 576.268, 691.473,
 806.651, 921.799, 1036.91, 1151.99, 1267.01, 1382., 1496.92}
```

Out[6]=
```
{7.203 × 10^-8, 6.91 × 10^-7, 2.497 × 10^-6, 6.126 × 10^-6, 0.00001221, 0.00002137,
 0.00003422, 0.00005134, 0.0000733, 0.0001006, 0.0001339, 0.0001735, 0.00022}
```

In[10]:=
```
columnDensities = 1.94 × 10^3 (frequencyList["CO"][[1 ;; 13]]^2)/(EinsteinAList["CO"][[1 ;; 13]]) data
```

Out[10]= {2.21846×10^{15}, 2.16211×10^{15}, 1.27833×10^{15}, 5.99428×10^{14}, 2.21133×10^{14}, 6.21137×10^{13}, 1.40657×10^{13}, 2.88847×10^{12}, 5.44943×10^{11}, 9.39212×10^{10}, 1.50344×10^{10}, 2.2509×10^{9}, 3.26033×10^{8}}

In[11]:=
```
Total[columnDensities]
```

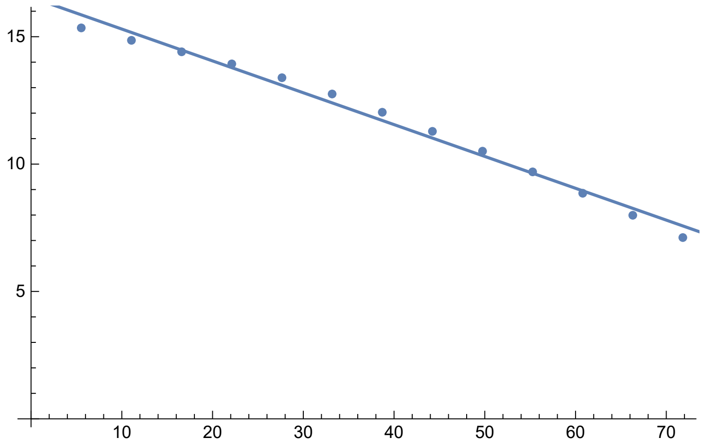
Out[11]= 6.55918×10^{15}

In[72]:=
```
statWeights = 2 (Range[13] - 1) + 1
```

Out[72]= {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25}

In[92]:=
```
Show[{ListPlot[Transpose[{energiesK, Log10@ columnDensities/statWeights}]],
  Plot[-0.125 x + 16.55, {x, 0, 100}]}]
```

Out[92]=



In[90]:=
```
FindFit[Transpose[{energiesK, Log10@ columnDensities/statWeights}], m x + b, {m, b}, x]
```

Out[90]= {m → -0.12519, b → 16.549}

Molecular Line Ratio Analysis

Determining physical parameters in a molecular cloud:
Two astronomical sources are observed in the spectral line emission of several CO Isotopomeres.
The measured data is summarized as follows:

In[165]:=
```
TableForm[{{"Source A", "", ""}, {"CO J=1-0", 23., 11.5}, {"13CO J=1-0", 0.46, 0.23},
  {"Source B", "", ""}, {"CO J=1-0", 32., 16.}, {"13CO J=1-0", 5.33, 2.67}},
 TableHeadings → {None, {"spectral line",
    "integr. line intensity\n∫T_Bdv (K km/s", "line intensity\nT_B(K)"}}]
```

Out[165]//TableForm=

spectral line	integr. line intensity $\int T_B dv$ (K km/s	line intensity T_B(K)
Source A		
CO J=1-0	23.	11.5
13CO J=1-0	0.46	0.23
Source B		
CO J=1-0	32.	16.
13CO J=1-0	5.33	2.67

All spectral lines are Gaussian with the same line width and 12CO is 50 times mote abundant than 13CO.

Estimate Optical depth

Estimate which of the line is optically thin. Use the detection equation with

$$T_B = (J_\nu(T_{ex}) - J_\nu(T_{bg}))(1 - \text{Exp}(-\tau_\nu)), \quad J_\nu = \frac{h\nu}{k}\frac{1}{\text{Exp}\left[\frac{h\nu}{kT}\right] - 1} \qquad (19)$$

to determine the optical depths and the excitation temperatures from the line ratios of the respective sources. Assume $T_{ex}(12\,CO) = T_{ex}(13\,CO)$, $\nu_{12} = \nu_{13}$, and $T_{bg} = 0$.

Source A: $\frac{T_B{}^{12}}{T_B{}^{13}} = \frac{11.5}{0.23} = 50.$ → both lines optically thin

Source A: $\frac{T_B{}^{12}}{T_B{}^{13}} = \frac{16}{2.67} = 5.99251$ → 12CO opt. thick, 13CO opt thin

Source A: both lines opt thin → $1 - \text{Exp}[-\tau] \approx \tau$

$$T_B = (J_\nu(T_{ex}) - J_\nu(T_{bg}))(1 - \text{Exp}(-\tau_\nu)) \approx J_\nu(T_{ex})\,\tau_\nu$$

$$\frac{T_B{}^{12}}{T_B{}^{13}} = \frac{J_\nu(T_{ex}{}^{12})\,\tau_{12}}{J_\nu(T_{ex}{}^{13})\,\tau_{13}} = \frac{\tau_{12}}{\tau_{13}} = \frac{50}{1}$$

$$\frac{T_B{}^{12}}{T_B{}^{13}} = \frac{50}{1} = \frac{\tau_{12}}{\tau_{13}}$$

because $\tau_{13} = \alpha\,\tau_{12} = \frac{1}{50}\,\tau_{12}$

here $\tau_{12/13}$ can not be determined as well as T_{ex}

Source B: 12CO opt thick, 13CO opt thin

$$\frac{T_B{}^{12}}{T_B{}^{13}} = \frac{1 - \text{Exp}[-\tau_{12}]}{1 - \text{Exp}[-\tau_{13}]} = \frac{1}{\tau_{13}} = \frac{1}{\alpha\,\tau_{12}}$$

$$\frac{T_B{}^{12}}{T_B{}^{13}} = 6 = \frac{1}{\alpha\,\tau_{12}} \Rightarrow \tau_{12} = \frac{50}{6} = 8.3$$

Computing T_{ex}

$$T_B{}^{12} = J_\nu(T_{ex}{}^{12})\,(1 - \text{Exp}[-\tau_{12}]) \approx J_\nu(T_{ex}{}^{12}) = \frac{h\nu}{k}\frac{1}{\text{Exp}\left[\frac{h\nu}{k\,T_{ex}}\right] - 1}$$

$$T_{ex} = \frac{h\,\nu}{k} \frac{1}{\text{Log}\left[\frac{h\,\nu}{k\,T_B{}^{12}} + 1\right]} \quad \text{for 12 CO 1 - 0}: \frac{h\,\nu}{k} = 5.53\ \text{K}$$

$$T_{ex} = \frac{5.53\ \text{K}}{\text{Log}\left[\frac{5.53\ \text{K}}{T_B{}^{12}} + 1\right]} = 18.6\ \text{K}$$

Column density from opt thin lines

$$N_{tot} = \frac{3\,k^2\,T_{ex}}{8\,\pi^3\,J\,\mu^2\,h\,\nu\,B}\,\text{Exp}\left[\frac{h\,B\,J\,(J+1)}{k\,T_{ex}}\right] \int T_B\,dv \qquad (20)$$

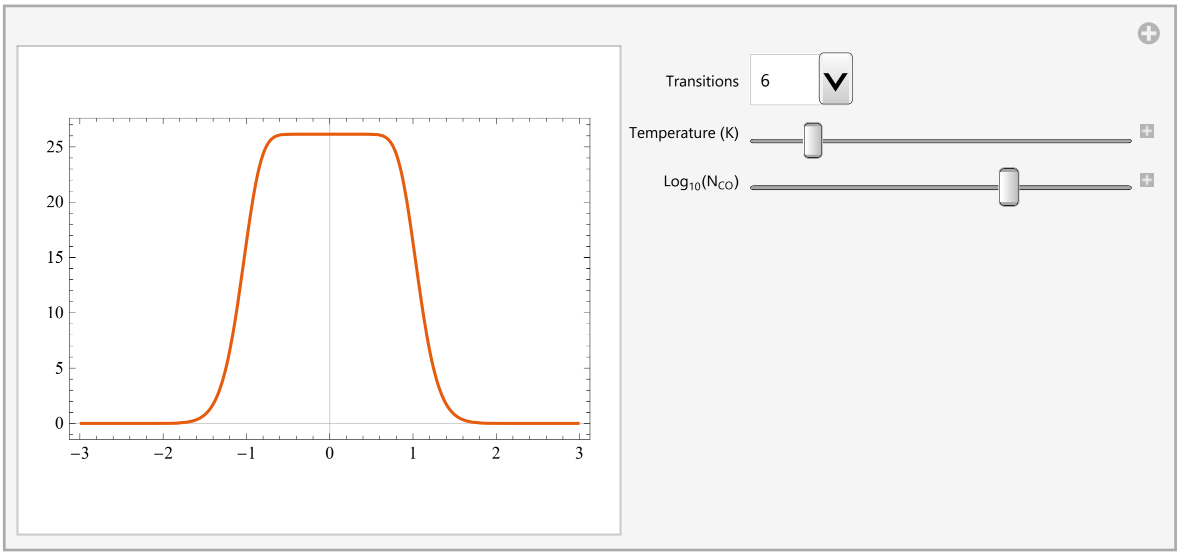
For 13CO 1-0: $T_{ex} = 18.6\,K$, $\int T_B\,dv = 5.33\ \text{K km s}^{-1}$, $\mu = 0.112\,D = 0.112 \times 10^{-18}\ \text{cm}^{5/2}\,g^{1/2}\,s$, B=55.1 GHz

$N_{tot} = 6 \times 10^{15}\ \text{cm}^{-2}$

LTE Radiative Transfer

```
In[169]:= Manipulate[Plot[LTEBrightnessTemperature[{"CO", i}, {t, 10^col, f}], {f, -3, 3},
   PlotRange → Automatic, PlotTheme → "Scientific"], {{i, 1, "Transitions"}, Range[20]},
  {{t, 10, "Temperature (K)"}, 3, 300}, {{col, 16, "Log10(NCO)"}, 12, 20}]
```

Out[169]=

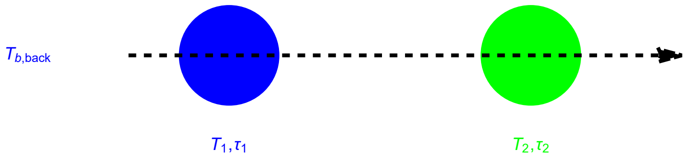


Self Absorption

A common case in studies of the interstellar medium is foreground absorption, i.e. radiation emitted by hot gas passes through a colder foreground cloud on its way to the observer:

In[170]:=
```
fig2 = Graphics[{Blue, Disk[{0, 0}, 1], Text["T₁,τ₁", {0, -1.8}],
   Text["Tb,back", {-4, 0}], Green, Disk[{6, 0}, 1], Text["T₂,τ₂", {6, -1.8}],
   Thick, Black, Dashed, Arrowheads[Medium], Arrow[{{-2, 0}, {9, 0}}]}]
```

Out[170]=



Blue (warm) cloud with a green (cool) foreground component.

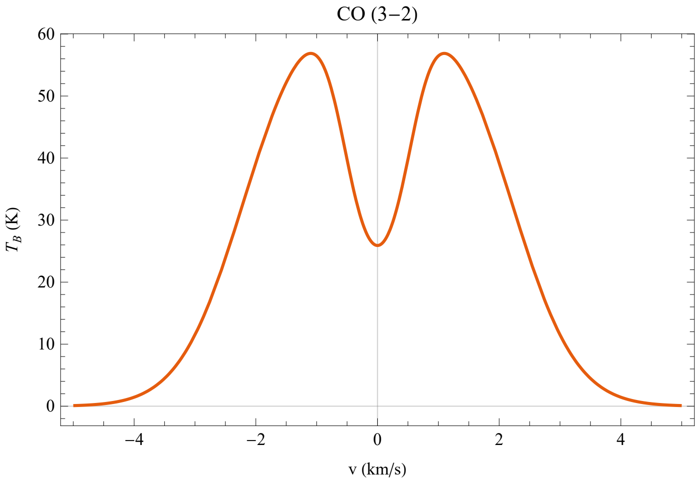
The total observed emission is :

$$T_b = T_{b,\mathrm{back}} \times \exp[-\tau_1] \times \exp[-\tau_2] + \frac{h\nu}{k}\Delta\nu \frac{1}{\exp\left(\frac{h\nu}{kT_1}\right)-1} \times (1-\exp[-\tau_1])\exp[-\tau_2] + \frac{h\nu}{k}\Delta\nu \frac{1}{\exp\left(\frac{h\nu}{kT_2}\right)-1} \times (1-\exp[-\tau_2]) - T_{b,\mathrm{back}}$$

In[171]:=
```
h = 6.62607 * 10^-27;
k = 1.38065 * 10^-16;
tMainBeamLTE2Component[v_, {v0b_, Δvb_, τ0b_, Tb_}, {v0f_, Δvf_, τ0f_, Tf_}, jUp_] :=
 With[{
   τf = τ0f Exp[-(v - v0f)^2 / (2. (Δvf / (2 Sqrt[2 Log[2.]]))^2)], τb = τ0b Exp[-(v - v0b)^2 / (2. (Δvb / (2 Sqrt[2 Log[2.]]))^2)],
   Tback =
    h/k Δvb transitionFrequency["CO", jUp] (Exp[h transitionFrequency["CO", jUp] / (k Tb)] - 1)^-1,
   Tfore = h/k Δvf transitionFrequency["CO", jUp]
     (Exp[h transitionFrequency["CO", jUp] / (k Tf)] - 1)^-1, Tcmb = h/k Δvb
      transitionFrequency["CO", jUp] (Exp[h transitionFrequency["CO", jUp] / (k 2.73)] - 1)^-1},
  Tback (1 - Exp[-τb]) Exp[-τf] + Tfore (1 - Exp[-τf]) + Tcmb (Exp[-τb] Exp[-τf] - 1)]
```

In[174]:=
```
v0b = 0; Δvb = 3; τ0b = 3; Tb = 30;
v0f = 0; Δvf = 1; τ0f = 1; Tf = 10;
jUp = 3;
Plot[tMainBeamLTE2Component[v, {v0b, Δvb, τ0b, Tb}, {v0f, Δvf, τ0f, Tf}, jUp],
 {v, -5, 5}, PlotTheme → "Scientific", ImageSize → Medium,
 FrameLabel → {"v (km/s)", "T_B (K)"}, PlotLabel → Row[{"CO (", jUp, "-", jUp - 1, ")"}]]
```

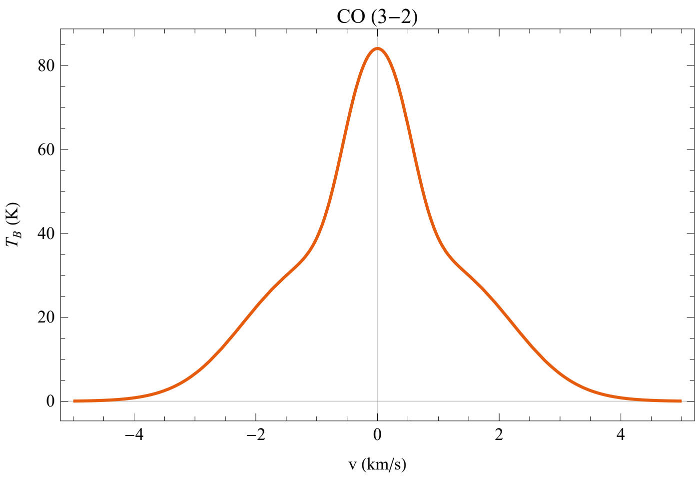
Out[177]=



If the foreground would be warmer than the background it would emit instead of absorb:

In[191]:=
```
v0b = 0; Δvb = 3; τ0b = 3; Tb = 20;
v0f = 0; Δvf = 1; τ0f = 1; Tf = 120;
jUp = 3;
Plot[tMainBeamLTE2Component[v, {v0b, Δvb, τ0b, Tb}, {v0f, Δvf, τ0f, Tf}, jUp],
 {v, -5, 5}, PlotTheme → "Scientific", ImageSize → Medium,
 FrameLabel → {"v (km/s)", "T_B (K)"}, PlotLabel → Row[{"CO (", jUp, "-", jUp - 1, ")"}]]
```

Out[194]=



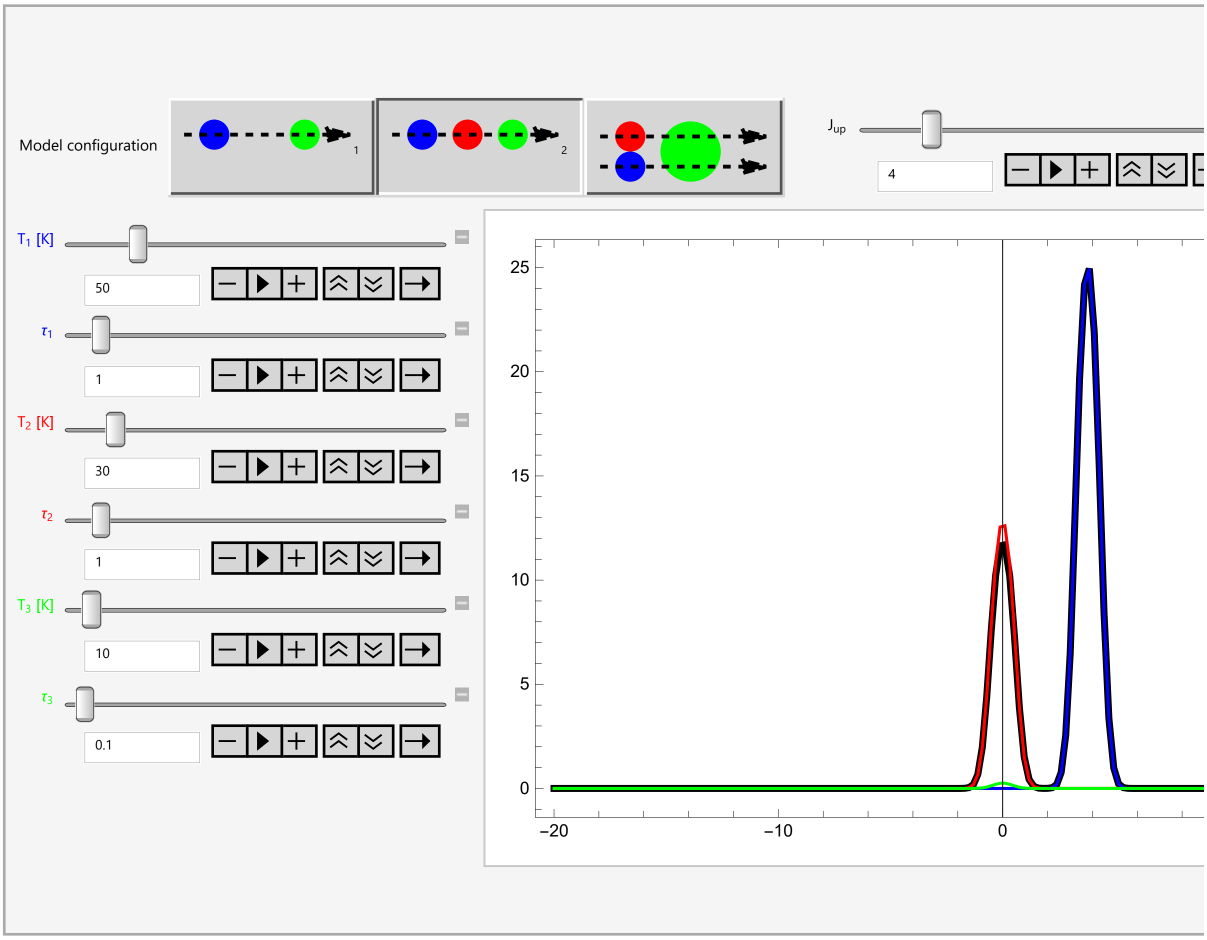
More complexity

In[178]:=
```
fig3 = Graphics[{Blue, Disk[{0, 0}, 1], Red, Disk[{3, 0}, 1], Green, Disk[{6, 0}, 1],
    Thick, Black, Dashed, Arrowheads[Medium], Arrow[{{-2, 0}, {9, 0}}]}];
fig2 = Graphics[{Blue, Disk[{0, 0}, 1], Green, Disk[{6, 0}, 1], Thick,
    Black, Dashed, Arrowheads[Medium], Arrow[{{-2, 0}, {9, 0}}]}];
fig2plus1 = Graphics[{Blue, Disk[{0, 0}, 1], Red, Disk[{0, 2}, 1], Green,
    Disk[{4, 1}, 2], Thick, Black, Dashed, Arrowheads[Medium],
    Arrow[{{-2, 0}, {9, 0}}], Arrow[{{-2, 2}, {9, 2}}]}];
```

```
h = 6.62607 * 10^-27;
k = 1.38065 * 10^-16;
cl = 2.99792458 × 10^10;
tMainBeamLTE2Component[v_,
   {v0b_, Δvb_, τ0b_, Tb_}, {v0f_, Δvf_, τ0f_, Tf_}, jUp_, B_] :=
  With[{τf = τ0f Exp[-(v - v0f)^2/(2. (Δvf/(2 Sqrt[2 Log[2.]]))^2)], τb = τ0b Exp[-(v - v0b)^2/(2. (Δvb/(2 Sqrt[2 Log[2.]]))^2)],
    Tback = h/k Δvb transitionFrequency["CO", jUp]
      (Exp[(h transitionFrequency["CO", jUp])/(k Tb)] - 1)^-1,
    Tfore = h/k Δvf transitionFrequency["CO", jUp]
      (Exp[(h transitionFrequency["CO", jUp])/(k Tf)] - 1)^-1, Tcmb = h/k Δvb
       transitionFrequency["CO", jUp] (Exp[(h transitionFrequency["CO", jUp])/(k 2.73)] - 1)^-1},
   Tback (1 - Exp[-τb]) Exp[-τf] + Tfore (1 - Exp[-τf]) + Tcmb (Exp[-τb] Exp[-τf] - 1)];
tMainBeamLTE1Component[v_, {v0b_, Δvb_, τ0b_, Tb_}, jUp_, B_] :=
 With[{τb = τ0b Exp[-(v - v0b)^2/(2. (Δvb/(2 Sqrt[2 Log[2.]]))^2)],
   Tback =
    h/k Δvb transitionFrequency["CO", jUp] (Exp[(h transitionFrequency["CO", jUp])/(k Tb)] - 1)^-1,
   Tcmb = h/k Δvb transitionFrequency["CO", jUp]
     (Exp[(h transitionFrequency["CO", jUp])/(k 2.73)] - 1)^-1},
  Tback (1 - Exp[-τb]) + Tcmb (Exp[-τb] - 1)]
tMainBeamLTE3Component[v_, {v01_, Δv1_, τ01_, T1_},
  {v02_, Δv2_, τ02_, T2_}, {v03_, Δv3_, τ03_, T3_}, jUp_, B_] :=
 (* All components line up. Comp 1 travels thropugh 2 and 3, 2 travels through 3*)
 With[{τ3 = τ03 Exp[-(v - v03)^2/(2. (Δv3/(2 Sqrt[2 Log[2.]]))^2)],
   τ1 = τ01 Exp[-(v - v01)^2/(2. (Δv1/(2 Sqrt[2 Log[2.]]))^2)], τ2 = τ02 Exp[-(v - v02)^2/(2. (Δv2/(2 Sqrt[2 Log[2.]]))^2)],
   Tr1 = h/k Δv1 transitionFrequency["CO", jUp]
     (Exp[(h transitionFrequency["CO", jUp])/(k T1)] - 1)^-1, Tr2 = h/k Δv2
      transitionFrequency["CO", jUp] (Exp[(h transitionFrequency["CO", jUp])/(k T2)] - 1)^-1, Tr3 =
    h/k Δv3 transitionFrequency["CO", jUp] (Exp[(h transitionFrequency["CO", jUp])/(k T3)] - 1)^-1,
   Tcmb = h/k Δv1 transitionFrequency["CO", jUp]
```

```
      (Exp[(h transitionFrequency["CO", jUp])/(k 2.73)] - 1)^-1},
   Tr1 (1 - Exp[-τ1]) Exp[-τ2] Exp[-τ3] + Tr2 (1 - Exp[-τ2]) Exp[-τ3] +
    Tr3 (1 - Exp[-τ3]) + Tcmb (Exp[-τ1] Exp[-τ2] Exp[-τ3] - 1)]
tMainBeamLTE2Plus1Component[v_, {v01_, Δv1_, τ01_, T1_},
   {v02_, Δv2_, τ02_, T2_}, {v03_, Δv3_, τ03_, T3_}, jUp_, B_] :=
  (* Not all components line up. Comp 1 travels thropugh 3, 2 travels through 3*)
  With[{τ3 = τ03 Exp[-(v - v03)^2/(2. (Δv3/(2 Sqrt[2 Log[2.]]))^2)],
    τ1 = τ01 Exp[-(v - v01)^2/(2. (Δv1/(2 Sqrt[2 Log[2.]]))^2)], τ2 = τ02 Exp[-(v - v02)^2/(2. (Δv2/(2 Sqrt[2 Log[2.]]))^2)],
    Tr1 = h/k Δv1 transitionFrequency["CO", jUp]
      (Exp[(h transitionFrequency["CO", jUp])/(k T1)] - 1)^-1, Tr2 = h/k Δv2
     transitionFrequency["CO", jUp] (Exp[(h transitionFrequency["CO", jUp])/(k T2)] - 1)^-1, Tr3 =
    h/k Δv3 transitionFrequency["CO", jUp] (Exp[(h transitionFrequency["CO", jUp])/(k T3)] - 1)^-1,
    Tcmb = h/k Δv1 transitionFrequency["CO", jUp]
      (Exp[(h transitionFrequency["CO", jUp])/(k 2.73)] - 1)^-1},
   (Tr1 (1 - Exp[-τ1]) + Tr2 (1 - Exp[-τ2])) Exp[-τ3] + Tr3 (1 - Exp[-τ3]) +
    Tcmb (1/2 (Exp[-τ1] + Exp[-τ2]) Exp[-τ3] - 1)]

In[189]:= SetSystemOptions["CheckMachineUnderflow" → False]

Out[189]= CheckMachineUnderflow → False
```

Out[190]=


Model configuration
1
2
J_{up}
4
T_1 [K]
50
τ_1
1
T_2 [K]
30
τ_2
1
T_3 [K]
10
τ_3
0.1
25
20
15
10
5
0
−20
−10
0